Name: ____________________ Date: __________ Period: ____ Key ⑤

BC Calculus Unit 9 Parametric & Polar Test Review WS #2

Calculators Allowed: Show all work that lead to your answer to earn full credit.

1. What is the length of the curved defined by the parametric equations $x(t) = 9\cos t$ and $y(t) = 9\sin t$ for the interval $0 \le t \le 2\pi$?

$\frac{dx}{dt} = -9\sin(t)$

$\frac{dy}{dt} = 9\cos(t)$

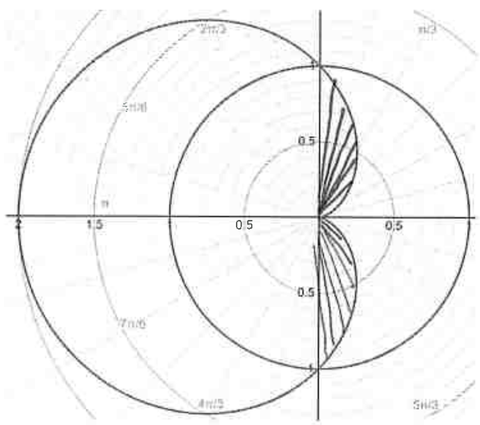
$L = \int_0^{2\pi} \sqrt{(-9\sin t)^2 + (9\cos t)^2}\,dt$

$= \int_0^{2\pi} \sqrt{81\sin^2 t + 81\cos^2 t}\,dt$

$\int \sqrt{81(\sin^2 t + \cos^2 t)}\,dt \rightarrow \int 9\sqrt{\sin^2 t + \cos^2 t}\,dt$

$9\int \sqrt{1}\,dt \rightarrow 9t\Big]_0^{2\pi} = 9(2\pi) - 0 = \boxed{18\pi}$

2. **Calculator active.** Find the area of the region inside the circle $r = 1$ and outside the cardiod $r = 1 - \cos\theta$.



Area of Region = Semicircle $-$ 2(half-loop)

$= \frac{1}{2}\pi r^2 - 2\left[\frac{1}{2}\int_0^{\pi/2} [1-\cos\theta]^2\,d\theta\right]$

$= \frac{\pi}{2}(1)^2 - 0.35619$

$\approx \boxed{1.215}$

3. If $x(t) = 2t^3$ and $y(t) = t^3 - t$, what is $\frac{d^2y}{dx^2}$ in terms of t?

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} \rightarrow \frac{3t^2-1}{6t^2} \rightarrow \frac{3t^2}{6t^2} - \frac{1}{6t^2} \rightarrow \frac{1}{2} - \frac{1}{6}t^{-2}$

$\frac{d^2y}{dx^2} = \frac{\left(\frac{dy}{dx}\right)'}{dx/dt} \rightarrow \frac{0 + 2\cdot\frac{1}{6}t^{-3}}{6t^2} \rightarrow \frac{\frac{1}{3t^3}}{6t^2} \rightarrow \frac{1}{3t^3}\cdot\frac{1}{6t^2} = \boxed{\frac{1}{18t^5}}$

4. The position of a remote-controlled vehicle moving along a flat surface at time t is given by $(x(t), y(t))$, with velocity vector $v(t) = \langle 3t^2, 2t\rangle$ for $0 \le t \le 3$. Both $x(t)$ and $y(t)$ are measured in meters, and time t is in seconds. When $t = 0$, the remote-controlled vehicle is at the point $(1, 2)$.

a. Find the acceleration vector of the remote-controlled vehicle when $t = 2$.

$a(t) = \langle 6t, 2\rangle$ $\boxed{a(2) = \langle 12, 2\rangle}$

b. Find the position of the remote-controlled vehicle when $t = 3$.

$x = \int 3t^2\,dt$

$x = \frac{3t^3}{3} + C_1$

$x = t^3 + C_1$

$1 = 0^3 + C_1$

$1 = C_1$

$y = \int 2t\,dt$

$y = \frac{2t^2}{2} + C_2$

$y = t^2 + C_2$

$2 = 0 + C_2$

$2 = C_2$

$x(t) = \langle t^3+1, t^2+2\rangle$

$x(3) = \langle 27+1, 3^2+2\rangle$

$\boxed{x(3) = \langle 28, 11\rangle}$

5. Which of the following gives the length of the path described by the parametric equations $x = 2e^{3t}$ and $y = 3t^2 + t$ from $0 \le t \le 1$?

A. $\int_0^1 \sqrt{12e^{6t} + (6t+1)^2}\, dt$

B. $\int_0^1 \sqrt{4e^{6t} + (6t+1)^2}\, dt$

C. $\int_0^1 \sqrt{4e^{6t} + 9t^4 + t^2}\, dt$

D. $\boxed{\int_0^1 \sqrt{36e^{6t} + (6t+1)^2}\, dt}$

$$*L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = 2e^{3t} \cdot 3 = 6e^{3t} \qquad \frac{dy}{dt} = 6t+1$$

$$L = \int_0^1 \sqrt{(6e^{3t})^2 + (6t+1)^2}\, dt$$

6. **Calculator active.** A polar curve is given by $r = \frac{5}{3 - \sin\theta}$. What angle θ corresponds on the curve with a y-coordinate of -1?

$[0, 2\pi]$

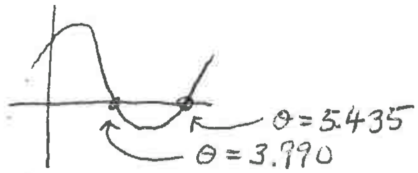
$$*y = r\sin\theta$$

$$y = \frac{5}{3-\sin\theta} \cdot \sin\theta$$

$$-1 = \frac{5\sin\theta}{3-\sin\theta}$$

$$0 = \frac{5\sin\theta}{3-\sin\theta}$$

* Graph this and look for x-intercepts



$$\boxed{\begin{array}{l}\theta = 3.990 \text{ or} \\ \theta = 5.435\end{array}}$$

7. If f is a vector-valued function defined by $\langle te^t,\ 2t^2e^t \rangle$ then $f''(1) = ?$

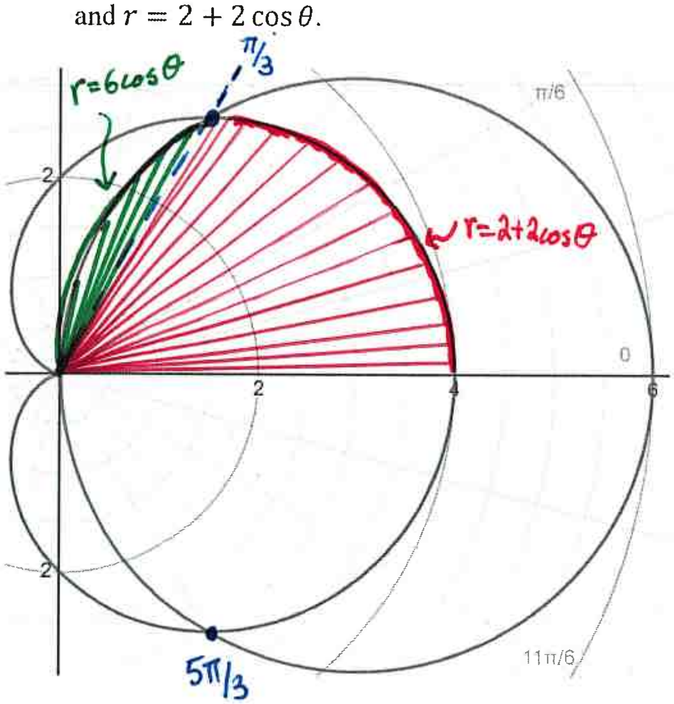
$$f'(t) = \langle 1e^t + te^t,\ 4te^t + 2t^2e^t \rangle$$

$$f''(t) = \langle e^t + 1e^t + te^t,\ 4e^t + 4te^t + 4te^t + 2t^2e^t \rangle$$

$$f''(1) = \langle e + e + e,\ 4e + 4e + 4e + 2e \rangle$$

$$\boxed{f''(1) = \langle 3e, 14e \rangle}$$

8. **Calculator active.** Find the area of the region common to the two regions bounded by the curves $r = 6\cos\theta$ and $r = 2 + 2\cos\theta$.



*find intersection:

$$6\cos\theta = 2 + 2\cos\theta$$

$$4\cos\theta = 2$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \pi/3,\ 5\pi/3$$

$$2\left[\frac{1}{2}\int_0^{\pi/3} (2 + 2\cos\theta)^2\, d\theta + \frac{1}{2}\int_{\pi/3}^{\pi} (6\cos\theta)^2\, d\theta\right]$$

$$14.0774 + 26.6437$$

$$= \boxed{40.721}$$

9. Find the vector-valued function $f(t)$ that satisfies the initial conditions $f(0) = \langle 3, 0\rangle$, and $f'(t) = \langle 4\sin\frac{t}{2}, -2\cos 2t\rangle$.

$x(t) = \int 4\sin\left(\frac{1}{2}t\right)dt$

$u = \frac{1}{2}t$ $\quad \frac{du}{dt} = \frac{1}{2}$ $\quad dt = 2du$ $\quad \int 4\sin u \cdot 2du$ $\quad \int 8\sin u\, du$

$x(t) = -8\cos\left(\frac{t}{2}\right) + C$ ← plug in (0,3)

$3 = -8\cos(0) + C$

$3 = -8 + C$

$-11 = C$

$x(t) = -8\cos(t/2) - 11$

$y(t) = \int -2\cos 2t\, dt$ $\quad u = 2t$ $\quad \frac{du}{dt} = 2$ $\quad dt = \frac{du}{2}$

$y(t) = \int -2\cos u \cdot \frac{du}{2}$

$y(t) = -\sin(2t) + C$ ← plug in (0,0)

$0 = 0 + C \quad C = 0$

$\boxed{f(t) = \langle -8\cos(\tfrac{t}{2}) + 11, -\sin(2t)\rangle}$

10. If $x = 7\cos\theta$ and $y = 7\sin\theta$, find the slope and the concavity at $\theta = \frac{\pi}{4}$.

$y'(\theta) = 7\cos\theta$

$x'(\theta) = -7\sin\theta$

$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \frac{y'(\pi/4)}{x'(\pi/4)} = \frac{7\cos(\pi/4)}{-7\sin(\pi/4)} = -1$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left[\frac{dy}{dx}\right]}{dx/d\theta} \to \frac{\frac{d}{d\theta}[-\cot\theta]}{-7\sin\theta} \to \frac{+\csc^2(\theta)}{-7\sin\theta} = -\frac{1}{7}\csc^3\theta$

$\frac{dy}{dx} = -\cot\theta$

$\left.\frac{d^2y}{dx^2}\right|_{\theta=\pi/4} = -\frac{1}{7}\left[\csc(\pi/4)\right]^3 \to -\frac{1}{7}\cdot\left(\frac{2}{\sqrt{2}}\right)^3 = \boxed{-\frac{2\sqrt{2}}{7}}$ $\boxed{\text{Concave down}}$

11. **Calculator active.** At time $t \geq 0$, a particle moving in the xy-plane has velocity vector given by $v(t) = \langle 9t^2, e^t\rangle$. If the particle is at point $(3, 4)$ at time $t = 0$, how far is the particle from the origin at time $t = 2$?

*final position = initial position + displacement

$x(b) = x(a) + \int_a^b v(t)dt$

$x(2) = x(0) + \int_0^2 x'(t)dt$

$x(2) = 3 + \int_0^2 9t^2 dt$

$= 3 + 24$

$x(2) = 27$

$y(2) = y(0) + \int_0^2 y'(t)dt$

$y(2) = 4 + \int_0^2 e^t dt$

$y(2) = 4 + 6.389$

$y(2) = 10.389$

At $t=2$, particle is at point $(27, 10.389)$

Distance from origin is $d = \sqrt{(27-0)^2 + (10.389-0)^2}$

$\boxed{d = 28.930}$

12. Find the slope of the tangent line to the polar curve $r = 2\cos\theta - 1$ at the point where $\theta = \frac{3\pi}{2}$.

$x(\theta) = r\cos\theta$

$x(\theta) = (2\cos\theta - 1)\cos\theta$

$x'(\theta) = (-2\sin\theta)\cos\theta + (2\cos\theta - 1)(-\sin\theta)$

$x'(\frac{3\pi}{2}) = (-2\sin\frac{3\pi}{2})\cos(3\pi/2) + (2\cos\frac{3\pi}{2} - 1)(-\sin\frac{3\pi}{2})$

$= (2)(0) + (0-1)(1)$

$x'(\frac{3\pi}{2}) = -1$

$y(\theta) = r\sin\theta$

$y(\theta) = (2\cos\theta - 1)\sin\theta$

$y'(\theta) = (-2\sin\theta)\sin\theta + (2\cos\theta - 1)(\cos\theta)$

$y'(\frac{3\pi}{2}) = -2\left[\sin(\frac{3\pi}{2})\right]^2 + (2\cos\frac{3\pi}{2} - 1)(\cos\frac{3\pi}{2})$

$= -2(-1)^2 + (0-1)(0)$

$y'(\frac{3\pi}{2}) = -2$

slope $= \frac{y'(3\pi/2)}{x'(3\pi/2)} = \frac{-2}{-1} = \boxed{2}$

13. Find the slope of the tangent line to the curve defined parametrically by $x(t) = 2\cos t$ and $y(t) = 3\sin^2 t$ at $t = \frac{\pi}{3}$.

$y(t) = 3[\sin(t)]^2$

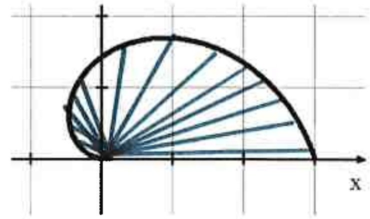
$y'(t) = 6[\sin(t)]\cdot\cos(t)$

$x'(t) = -2\sin(t)$

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} \to \frac{6\cancel{\sin t}\cos t}{-2\cancel{\sin t}} = -3\cos(t)$

$\left.\frac{dy}{dt}\right|_{t=\pi/3} = -3\cos(\pi/3) = -3\left(\frac{1}{2}\right) = \boxed{-\frac{3}{2}}$

14. **Calculator active.** The graph shows the polar curve $r = 3 - \theta$ for $0 \le \theta \le \pi$. What is the area of the region bounded by the curve and the x-axis?



$$A = \frac{1}{2}\int_a^b r^2 d\theta$$

$$A = \frac{1}{2}\int_0^{\pi} [3-\theta]^2 d\theta \approx \boxed{4.500}$$

15. At time t, $0 \le t \le 2\pi$, the position of a particle moving along a path in the xy-plane is given by the vector-valued function, $f(t) = \langle \cos 2t, \sin 4t \rangle$. Find the slope of the path of the particle at time $t = \frac{\pi}{4}$.

$$f'(t) = \langle -2\sin(2t), 4\cos(4t) \rangle$$

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \frac{y'(\pi/4)}{x'(\pi/4)} \rightarrow \frac{4\cos(4\cdot\pi/4)}{-2\sin(2\cdot\pi/4)} \rightarrow \frac{4\cos(\pi)}{-2\sin(\pi/2)} \rightarrow \frac{4(-1)}{-2(1)} \rightarrow \frac{-4}{-2} = \boxed{2}$$

16. Find an equation for the line tangent to the curve given by the parametric equations $x(t) = t^2 + 1$ and $y(t) = t^3 + t + 1$, when $t = 2$.

$x(2) = 2^2 + 1 = 5$

$y(2) = 2^3 + 2 + 1 = 11$

$x'(t) = 2t$

$y'(t) = 3t^2 + 1$

$\text{slope} = \frac{y'(2)}{x'(2)} = \frac{3(2)^2+1}{2(2)} = \frac{13}{4}$

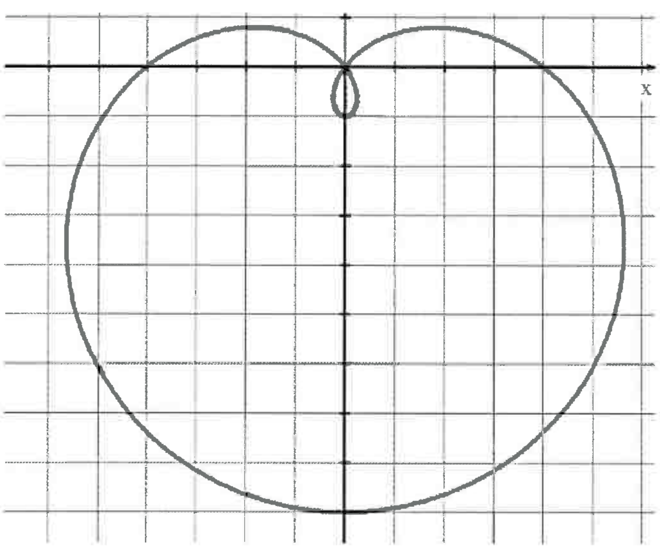
point: $(5, 11)$

slope: $m = 13/4$

$$\boxed{y - 11 = \frac{13}{4}(x-5) \quad \text{OR} \quad y = \frac{13}{4}x - \frac{21}{4}}$$

17. **Calculator active.** Find the total area enclose by the inner loop of the polar curve $r = 4 - 5\sin\theta$, shown in the figure.

*find polar zeros: $r = 4 - 5\sin\theta$, $0 = 4 - 5\sin\theta$ | $5\sin\theta = 4$, $\sin\theta = 4/5$ | $\theta \approx 0.927$



* $\sin\theta$ is also positive in 2^{nd} quadrant: since $\theta = 0.927$ is reference angle, $\theta_2 = \pi - 0.927 \approx 2.214$

$$\text{Area} = \frac{1}{2}\int_{0.927}^{2.214} [4 - 5\sin\theta]^2 d\theta \approx \boxed{0.340}$$