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Name: Key Date: ____________ Period: _____

BC Calculus Unit 9 Parametric and Polar Test Review WS #3

Calculators Allowed: Show all work that lead to your answer to earn full credit.

1) What is the slope of the tangent line to the curve defined parametrically by $x(t) = \sqrt{t}$ and $y(t) = \frac{1}{4}(t^2 - 4)$, $t \geq 0$ at the point (2,3)?

$x = \sqrt{t}$
$2 = \sqrt{t}$
$\underline{4 = t}$

$y(t) = \frac{1}{4}(t^2-4)$
$3 = \frac{1}{4}(t^2-4)$
$12 = t^2 - 4$
$16 = t^2$
$t = \pm 4 \rightarrow \underline{t=4}$

$y'(t) = \frac{1}{4}\cdot 2t = \frac{1}{2}t$
$x'(t) = \frac{1}{2}t^{-1/2}$

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} \rightarrow \frac{\frac{1}{2}t}{\frac{1}{2}t^{-1/2}} \rightarrow t^{3/2}$

$y(t) = \frac{1}{4}t^2 - 1$

$\left.\frac{dy}{dx}\right|_{t=4} = (4)^{3/2} = 2^3 = \boxed{8}$

2) If $x = \sin\theta$ and $y = 2\cos\theta$, what is $\frac{d^2y}{dx^2}$ in terms of θ?

$x'(\theta) = \cos\theta$
$y'(\theta) = -2\sin\theta$

$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{-2\sin\theta}{\cos\theta} = -2\tan\theta$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left[\frac{dy}{dx}\right]}{dx/d\theta} \rightarrow \frac{\frac{d}{d\theta}(-2\tan\theta)}{\cos\theta} \rightarrow \frac{-2\sec^2\theta}{\cos\theta}$

$\boxed{\frac{d^2y}{dx^2} = -2\sec^3\theta}$

3) Which of the following gives the length of the path described by the parametric equations $x = e^{2t}$ and $y = 1 - 2t$ from $0 \leq t \leq 3$?

A. $\int_0^3 \sqrt{4e^{2t} + 4}\, dt$ B. $\int_0^3 \sqrt{2e^{2t} + 2}\, dt$

$\boxed{\text{C. } \int_0^3 \sqrt{4e^{4t} + 4}\, dt}$ D. $\int_0^3 \sqrt{e^{4t} + 4}\, dt$

* $\text{Length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$\frac{dx}{dt} = 2e^{2t}$ $\frac{dy}{dt} = -2$

$\boxed{L = \int_0^3 \sqrt{(2e^{2t})^2 + (-2)^2}\, dt}$

4) The position of a particle moving in the xy-plane is defined by the vector-valued function, $f(t) = \langle t^3 - 9t^2 + 1,\ 2t^3 - 15t^2 - 36t + 1\rangle$. For what value of t is the particle at rest?

* set $v(t) = 0$ for both vertical and horizontal components
* find the t value(s) where the 2 components agree.

$f'(t) = \langle 3t^2 - 18t,\ 6t^2 - 30t - 36\rangle$

$3t^2 - 18t = 0$
$3t(t-6) = 0$
$t = 0,\ \underline{t = 6}$

$6t^2 - 30t - 36 = 0$
$6(t^2 - 5t - 6) = 0$
$6(t-6)(t+1) = 0$
$\underline{t = 6},\ t = -1$

$\boxed{t = 6}$

5) At time t, $0 \le t \le 2\pi$, the position of a particle moving along a path in the xy-plane is given by the vector-valued function, $f(t) = \langle e^{2t}\cos t,\ e^{2t}\sin t\rangle$. Find the slope of the path of the particle at time $t = \frac{\pi}{2}$.

$$x'(t) = 2e^{2t}\cos t + e^{2t}(-\sin t)$$
$$x'(\pi/2) = 2e^{\pi}\cos(\pi/2) + e^{\pi}(-\sin(\pi/2))$$
$$= 2e^{\pi}(0) + e^{\pi}(-1) = -e^{\pi}$$

$$y'(t) = 2e^{2t}\sin t + e^{2t}\cdot\cos t$$
$$y'(\pi/2) = 2e^{\pi}\sin(\pi/2) + e^{\pi}\cos(\pi/2)$$
$$= 2e^{\pi}(1) + e^{\pi}(0) = 2e^{\pi}$$

$$\left.\frac{dy}{dx}\right|_{t=\pi/2} = \frac{y'(\pi/2)}{x'(\pi/2)} = \frac{2e^{\pi}}{-e^{\pi}} = \boxed{-2}$$

6) **Calculator active.** At time $t \ge 0$, a particle moving in the xy-plane has a velocity vector given by $v(t) = \langle 2, 2^{-t^2}\rangle$. If the particle is at point $\left(1, \frac{1}{2}\right)$ at time $t = 0$, how far is the particle from the origin at time $t = 1$?

$$x(b) = x(a) + \int_a^b x'(t)\,dt$$
$$x(1) = x(0) + \int_0^1 2\,dt$$
$$x(1) = 1 + 2$$
$$x(1) = 3$$

$$y(b) = y(a) + \int_a^b y'(t)\,dt$$
$$y(1) = y(0) + \int_0^1 2^{-t^2}\,dt$$
$$y(1) = 1/2 + 0.8100$$
$$y(1) = 1.310$$

At $t=1$, particle is at point $(1, 1.31)$

Distance from origin is

$$d = \sqrt{(3-0)^2 + (1.31-0)^2}$$
$$\boxed{d = 3.274}$$

7) **Calculator active.** The position of a particle at time $t \ge 0$ is given by $x(t) = \frac{\sqrt{t+1}}{3}$ and $y(t) = t^2 + 1$. Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

$$\text{*Total Distance} = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$x(t) = \frac{1}{3}(t+1)^{1/2}$$
$$x'(t) = \frac{1}{3}\cdot\frac{1}{2}(t+1)^{-1/2}$$
$$x'(t) = \frac{1}{6\sqrt{t+1}}$$

$$y'(t) = 2t$$

$$\int_0^2 \sqrt{\left(\frac{1}{6\sqrt{t+1}}\right)^2 + (2t)^2}\,dt$$
$$= \boxed{4.023}$$

8) **Calculator active.** The velocity vector a particle moving in the xy-plane has components given by $\frac{dx}{dt} = \sin 2t$ and $\frac{dy}{dt} = e^{\cos t}$. At time $t = 2$, the position of the particle is $(3, 2)$. What is the x-coordinate of the position vector at time $t = 3$?

$$x(b) = x(a) + \int_a^b x'(t)\,dt$$
$$x(3) = x(2) + \int_2^3 \sin(2t)\,dt$$
$$x(3) = 3 + -0.8069$$

$$\boxed{x(3) = 2.193}$$

9) A particle moves along the polar curve $r = 4 - 2\cos\theta$ so that $\frac{d\theta}{dt} = 4$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$.

*think related rates steps

$r = 4 - 2\cos\theta$

$\frac{dr}{dt} = -2(-\sin\theta)\left(\frac{d\theta}{dt}\right)$

$\frac{dr}{dt} = 2\sin\left(\frac{\pi}{3}\right)\cdot 4$

$\frac{dr}{dt} = 8\sin(\pi/3)$

$\frac{dr}{dt} = 8\left(\frac{\sqrt{3}}{2}\right)$

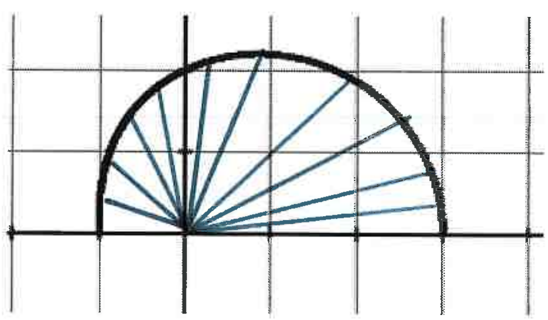
$\boxed{\frac{dr}{dt} = 4\sqrt{3}}$

10) **Calculator active.** For a certain polar curve $r = f(\theta)$, it is known that $\frac{dx}{d\theta} = 3\cos\theta - 3\theta\sin\theta$ and $\frac{dy}{d\theta} = 3(\sin\theta + \theta\cos\theta)$. What is the value of $\frac{d^2y}{dx^2}$ at $\theta = 3$?

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left[\frac{dy}{dx}\right]}{dx/d\theta} \rightarrow \frac{\frac{d}{d\theta}\left[\frac{3\sin\theta + 3\theta\cos\theta}{3\cos\theta - 3\theta\sin\theta}\right]}{3\cos\theta - 3\theta\sin\theta} \text{ at } \theta = 3 \rightarrow \frac{5.5067}{8.0632}$$

$\rightarrow \boxed{0.6829}$

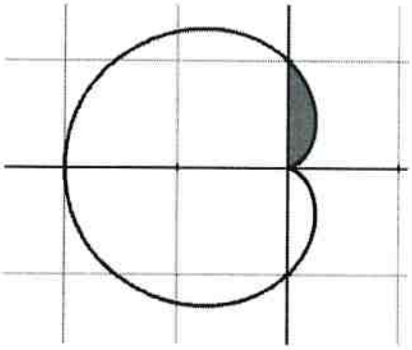
11) The graph to the right shows the polar curve $r = 2 + \cos\theta$ for $0 \le \theta \le \pi$. What is the area of the region bounded by the curve and the x-axis? 7.069



$A = \frac{1}{2}\int_0^{\pi} (2+\cos\theta)^2\, d\theta$

Area $\approx \boxed{7.069}$

12) Find the area of the shaded region for the polar curve $r = 1 - \cos\theta$. 0.178



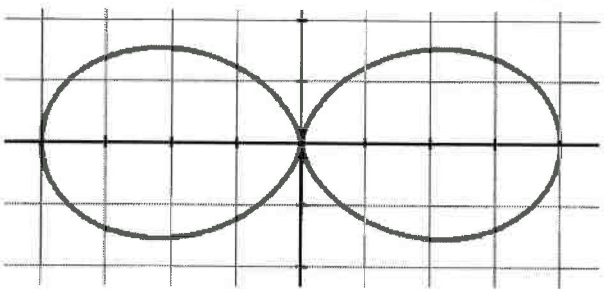
θ	r
0	0
$\pi/2$	1

$$\text{Area} = \frac{1}{2}\int_0^{\pi/2} [1-\cos\theta]^2 \, d\theta$$

$$\boxed{\text{Area} \approx 0.178}$$

13) Find the total area enclosed by the polar curve $r = 2 + 2\cos 2\theta$ shown in the figure

18.850

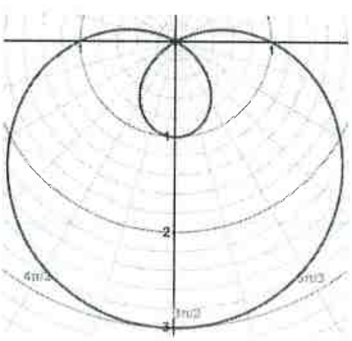


$$A = 2\left[\frac{1}{2}\int_0^{\pi} [2+2\cos(2\theta)]^2 d\theta\right]$$

OR

$$A = \frac{1}{2}\int_0^{2\pi} (2+2\cos 2\theta)^2 d\theta \approx \boxed{18.850}$$

14) Write do not solve, an integral expression that represents the area enclosed by the smaller loop of the polar curve $r = 1 - 2\sin\theta$.



* find polar zeros

$r = 1 - 2\sin\theta$

$0 = 1 - 2\sin\theta$

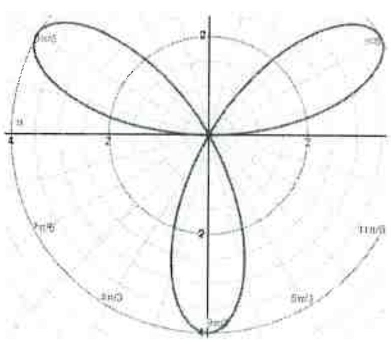
$2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

$\theta = \pi/6, \ 5\pi/6$

$$\boxed{\text{Area} = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1-2\sin\theta)^2 d\theta}$$

15) Find the limits of integration required to find the area of one petal of the polar graph $r = 4\sin 3\theta$ in the second quadrant.



r	θ
$2\pi/3$	$4\sin(3\cdot\frac{2\pi}{3}) = 0$
π	$4\sin(3\pi) = 0$

$$\boxed{\frac{2\pi}{3}, \pi}$$

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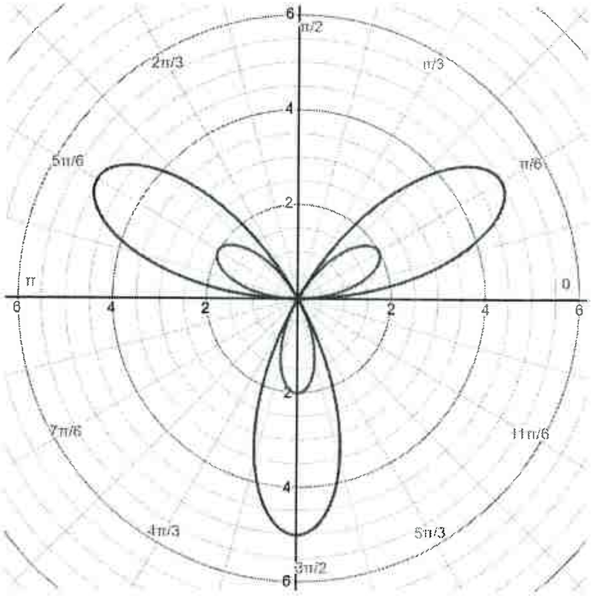
*polar zeros that form first quadrant petal are $\theta = 0$ and $\theta = \pi/3$

16) What is the total area between the polar curves $r = 2\sin 3\theta$ and $r = 5\sin 3\theta$.



Area of region for 1 petal = Area (larger petal) − Area (smaller petal)

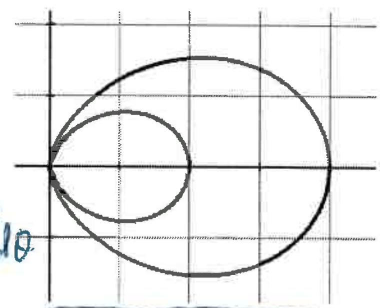
$$\text{Area} = \frac{1}{2}\int_0^{\pi/3} (5\sin 3\theta)^2 d\theta - \frac{1}{2}\int_0^{\pi/3} (2\sin 3\theta)^2 d\theta$$

Area = 5.4977

Area of all 3 petals $\rightarrow 3(5.4977) = \boxed{16.493}$

17)

The figure to the right shows the graphs of the polar curves $r = 2\cos^2\theta$ and $r = 4\cos^2\theta$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Which of the following integrals gives the area of the region bounded between the two polar curves?



A. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta \, d\theta$

B. $\boxed{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 6\cos^4\theta \, d\theta}$

C. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^4\theta \, d\theta$

D. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^2\theta \, d\theta$

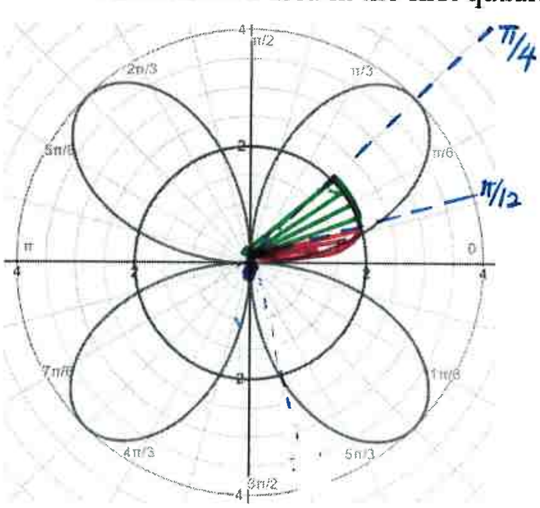
Area of Larger − Area of Smaller

$$\frac{1}{2}\int_{-\pi/2}^{\pi/2} [4\cos^2\theta]^2 d\theta - \frac{1}{2}\int_{-\pi/2}^{\pi/2} [2\cos^2\theta]^2 d\theta$$

$$\frac{1}{2}\int 16\cos^4\theta \, d\theta - \frac{1}{2}\int 4\cos^4\theta \, d\theta$$

$$8\int \cos^4\theta \, d\theta - 2\int \cos^4 d\theta \rightarrow \boxed{\int_{-\pi/2}^{\pi/2} 6\cos^4\theta \, d\theta}$$

18) Find the total area in the first quadrant of the common interior of $r = 4\sin 2\theta$ and $r = 2$.



*intersection: $4\sin 2\theta = 2$

$\sin 2\theta = \frac{2}{4}$

$\sin 2\theta = \frac{1}{2}$

$2\theta = \sin^{-1}\left(\frac{1}{2}\right)$

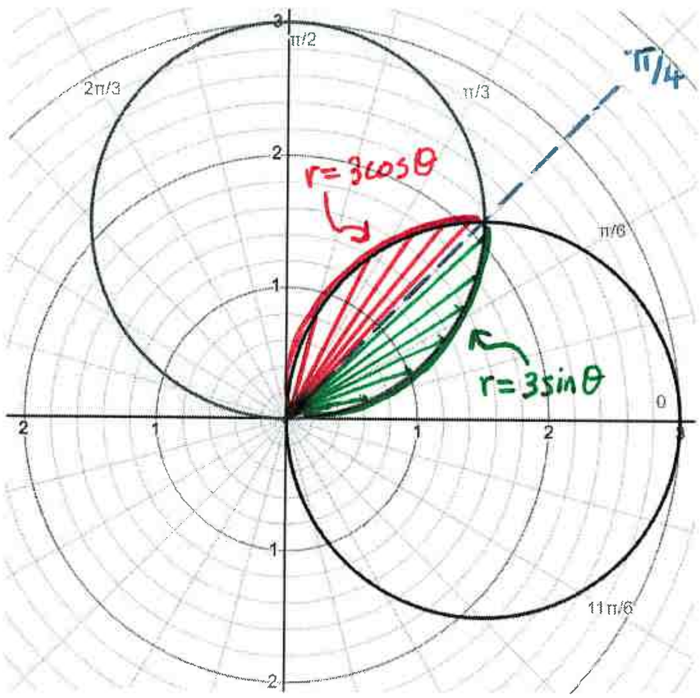
$2\theta = \frac{\pi}{6}$

$\theta = \frac{1}{2} \cdot \frac{\pi}{6} = \frac{\pi}{12}$

$$2\left[\frac{1}{2}\int_0^{\pi/12} (4\sin 2\theta)^2 d\theta + \frac{1}{2}\int_{\pi/12}^{\pi/4} (2)^2 d\theta\right]$$

$\boxed{\text{Area} = 2.457}$

19) Find the area of the common interior of the polar graphs $r = 3\cos\theta$ and $r = 3\sin\theta$.



*intersection:

$3\sin\theta = 3\cos\theta$

$\frac{3\sin\theta}{3\cos\theta} = \frac{3\cos\theta}{3\cos\theta}$

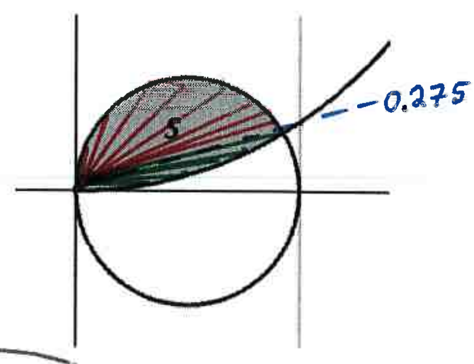
$\tan\theta = 1 \rightarrow \theta = \pi/4$

$$\frac{1}{2}\int_0^{\pi/4}[3\sin\theta]^2 d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2}[3\cos\theta]^2 d\theta$$

$\boxed{\text{Area} = 1.284}$

20)

Let S be the region in the 1st Quadrant bounded above by the graph of the polar curve $r = \cos\theta$ and bounded below by the graph of the polar curve $r = \frac{7}{2}\theta$, as shown in the figure. The two curves intersect when $\theta = 0.275$. What is the area of S?



$$\text{Area} = \frac{1}{2}\int_0^{0.275}\left(\frac{7}{2}\theta\right)^2 d\theta + \frac{1}{2}\int_{0.275}^{\pi/2}[\cos\theta]^2 d\theta$$

$$0.04246 + 0.258613 \approx \boxed{0.301}$$