2017

AB Calculus Ch. 3.4 Select HW Problems Key

Steps:
1) find $f''(x)$ (second derivative)
2) set $f''(x)=0$, find critical pts.
3) create sign line, evaluate concavity in intervals

Finding Points of Inflection In Exercises 15–30, find the points of inflection and discuss the concavity of the graph of the function.

15. $f(x) = x^3 - 6x^2 + 12x$

$f'(x) = 3x^2 - 12x + 12$

$f''(x) = 6x - 12$

$0 = 6(x-2)$

$x = 2$

Sign line: $\cap$ − | + $\cup$ at 2

Concave up: $(2, \infty)$ b/c $f''(x) > 0$

concave down: $(-\infty, 2)$ b/c $f''(x) < 0$

POI: $(2, \underline{8})$ b/c $f''(x)$ change signs

17. $f(x) = \frac{1}{2}x^4 + 2x^3$

$f'(x) = 4\cdot\frac{1}{2}x^3 + 6x^2 = 2x^3 + 6x^2$

$f''(x) = 6x^2 + 12x$

$0 = 6x(x+2)$

$x = 0, -2$

Sign line: $\cup$ + | − $\cap$ | + $\cup$ at −2, 0

Concave up $(-\infty, -2) \cup (0, \infty)$ b/c $f''(x) > 0$

Concave down $(-2, 0)$ b/c $f''(x) < 0$

POI: $(-2, \underline{-8})$ and $(0, \underline{0})$ b/c $f''(x)$ change signs

19. $f(x) = x(x-4)^3$ * apply product, chain rule

$f'(x) = 1\cdot(x-4)^3 + x\cdot 3(x-4)^2(1)$

$= (x-4)^2[x - 4 + 3x] = (x-4)^2(4x-4)$

$f''(x) = 2(x-4)\cdot(4x-4) + (x-4)^2(4)$

$= 2(x-4)\cdot 4(x-1) + (x-4)^2\cdot 4$

$= 4(x-4)[2(x-1) + x - 4] = 4(x-4)(3x-6)$ (with $2x-2+x-4$ written above the bracket)

$f''(x) = 12(x-4)(x-2)$

$0 = 12(x-4)(x-2)$

$x = 2, 4$

Sign line: $\cup$ + | − $\cap$ | + $\cup$ at 2, 4

concave up $(-\infty, 2) \cup (4, \infty)$

concave down $(2, 4)$

POI: $(2, \underline{-16})$, $(4, \underline{0})$

21. $f(x) = x\sqrt{x+3}$ * Apply product, chain rule

Domain: $[-3, \infty)$

$f(x) = x(x+3)^{1/2}$

$f'(x) = 1(x+3)^{1/2} + x\cdot\frac{1}{2}(x+3)^{-1/2}(1)$

$f'(x) = \sqrt{x+3} + \dfrac{x}{2\sqrt{x+3}} = \dfrac{2(x+3)+x}{2\sqrt{x+3}} = \dfrac{3x+6}{2\sqrt{x+3}}$

$f''(x) = \dfrac{3\cdot 2\sqrt{x+3} - (3x+6)\cdot 2\cdot\frac{1}{2}(x+3)^{-1/2}}{[2\sqrt{x+3}]^2}$

$= \dfrac{6\sqrt{x+3} - \frac{3x+6}{\sqrt{x+3}}}{4(x+3)} \cdot \dfrac{\sqrt{x+3}}{\sqrt{x+3}} = \dfrac{6(x+3) - (3x+6)}{4(x+3)^{3/2}}$ (with $6x+18-3x-6$ written above)

$f''(x) = \dfrac{3x+12}{4(x+3)^{3/2}}$

$x = -4, -3$ (−4 not in domain)

Sign line: × | + $\cup$ at −3

Concave up: $(-3, \infty)$

No POI.

23. $f(x) = \dfrac{4}{x^2+1} = 4(x^2+1)^{-1}$

$f'(x) = -4(x^2+1)^{-2}(2x) = \dfrac{-8x}{(x^2+1)^2}$

$f''(x) = \dfrac{-8(x^2+1)^2 - 8x\cdot 2(x^2+1)(2x)}{(x^2+1)^4}$

$= \dfrac{-8(x^2+1)[x^2+1-4x^2]}{(x^2+1)^4}$

$f''(x) = \dfrac{-8(1-3x^2)}{(x^2+1)^3}$

$x = \pm\frac{\sqrt{3}}{3}$

Sign line: $\cup$ + | − $\cap$ | + $\cup$ at $-\sqrt{3}/3$, $\sqrt{3}/3$

Concave up: $(-\infty, -\sqrt{3}/3) \cup (\sqrt{3}/3, \infty)$

Concave down: $(-\sqrt{3}/3, \sqrt{3}/3)$

POI: $(-\frac{\sqrt{3}}{3}, \underline{3})$ and $(\sqrt{3}/3, \underline{3})$

Using the Second Derivative Test In Exercises 31–42, find all relative extrema. Use the Second Derivative Test where applicable.

*steps:

1) find $f'(x)$, critical pts from $f'(x)=0$
2) plug critical pts into $f''(x)$
3) If $f''(x) > 0$, relative min
   If $f''(x) < 0$ rel. max
   If $f''(x) = 0$ inconclusive

33. $f(x) = x^3 - 3x^2 + 3$

$f'(x) = 3x^2 - 6x = 3x(x-2)$

$0 = 3x(x-2) \quad x = 0, 2$

$f''(x) = 6x - 6$

$f''(0) = 6(0) - 6 = -6 < 0$ concave down, rel. max

Rel. max at $(0, \underline{3})$

$f''(2) = 6(2) - 6 = 6 > 0$ rel. min

Rel. min at $(2, \underline{-1})$

35. $f(x) = x^4 - 4x^3 + 2$

$f'(x) = 4x^3 - 12x^2$

$f'(x) = 4x^2(x-3)$

$0 = 4x^2(x-3)$

$x = 0, 3$

$f''(x) = 12x^2 - 24x$

$f''(0) = 0$, inconclusive

$f''(3) = 16 > 0$, concave up, rel. min at $(3, \underline{-25})$

1st derivative test

↘ − ↘ − ↗ +, at 0 and 3

No relative extrema at $x = 0$.

37. $f(x) = x^{2/3} - 3$

$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$, $x = 0$

$f''(x) = -\frac{2}{9}x^{-4/3} = \frac{-2}{9x^{4/3}}$

$f''(0) =$ undefined, inconclusive

↘ − ↗ +, at 0

By 1st derivative test, relative min at $(0, \underline{-3})$ b/c $f'(x)$ changes from − to +

39. $f(x) = x + \frac{4}{x} = x + 4x^{-1}$

$f'(x) = 1 - 4x^{-2} = 1 - \frac{4}{x^2} = \frac{x^2-4}{x^2}$

$0 = \frac{x^2-4}{x^2}$ $\quad x = 0, 2, -2$

$f''(x) = \frac{2x(x^2) - (x^2-4)(2x)}{x^4} = \frac{2x^3 - 2x^3 + 8x}{x^4}$

$f''(x) = \frac{8}{x^3}$

$f''(0) =$ undefined, inconclusive

$f''(-2) = \frac{8}{-8} = -1 < 0$, concave down rel. max at $(-2, \underline{-4})$

$f''(2) = \frac{8}{8} = 1 > 0$, concave up, rel. min at $(2, \underline{4})$

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Think About It In Exercises 53–56, sketch the graph of a function f having the given characteristics.

53. $f(2) = f(4) = 0$ $\rightarrow (2, 0), (4, 0)$

$f'(x) < 0$ for $x < 3$

$f'(3)$ does not exist.

$f'(x) > 0$ for $x > 3$

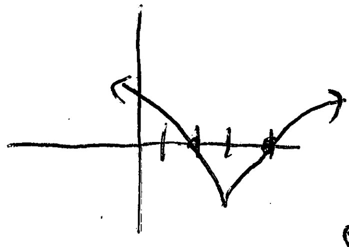
$f''(x) < 0, x \neq 3$ $\rightarrow$ concave down everywhere, except $x = 3$

$- \quad | \quad +$

3

$\cap \quad | \quad \cap$

3



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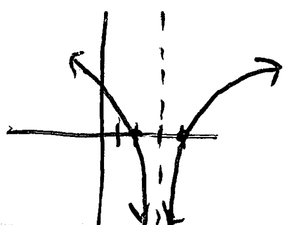

55. $f(2) = f(4) = 0$

$f'(x) > 0$ for $x < 3$

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$f'(x) < 0$ for $x > 3$

$f''(x) > 0, x \neq 3$

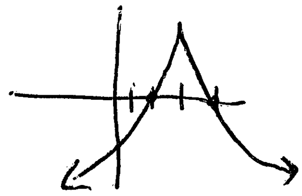
$+ \quad | \quad -$

3

$\cup \quad | \quad \cup$

$+ \quad | \quad +$

3



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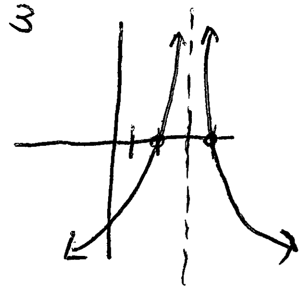

Finding a Cubic Function In Exercises 61 and 62, find a, b, c, and d such that the cubic

$f(x) = ax^3 + bx^2 + cx + d$

satisfies the given conditions.

$f'(x) = 3ax^2 + 2bx + c$

61. Relative maximum: $(3, 3)$ $\rightarrow f'(3) = 0 \rightarrow 0 = 3a(3)^2 + 2b(3) + c = \boxed{27a + 6b + c = 0}$

Relative minimum: $(5, 1)$ $\rightarrow f'(5) = 0 \rightarrow \cancel{0 = 3a(5)^2 + 2b(5) + c}$ $\cancel{75a + 10b + c = 0}$

Inflection point: $(4, 2)$ $\rightarrow f''(4) = 0 \rightarrow 0 = 6a(4) + 2b$

$0 = 24a + 2b$

$\cancel{-27a - 6b - c = 0}$

$\cancel{75a + 10b + c = 0}$

$\cancel{48a + 4b = 0}$

$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

$f(3) = a(3)^3 + b(3)^2 + c(3) + d = 27a + 9b + 3c + d = 3$

$f(5) = a(5)^3 + b(5)^2 + c(5) + d = 125a + 25b + 5c + d = 1$

$-27a - 9b - 3c - d = -3$

$125a + 25b + 5c + d = 1$

$\boxed{98a + 16b + 2c = -2}$

$-2(27a + 6b + c = 0)$

$98a + 16b + 2c = -2$

$-54a - 12b - 2c = 0$

$98a + 16b + 2c = -2$

$44a + 4b = -2$

$22a + 2b = -1$

$0 = 24a + 2b$

$-1 = 22a + 2b$

$0 = 24a + 2b$

$1 = -22a - 2b$

$1 = 2a$ $\boxed{a = 1/2}$

$f''(4) = 0$

$0 = 6(1/2)(4) + 2b$

$0 = 12 + 2b$

$\boxed{b = -6}$

$\boxed{c = 45/2}$

$\boxed{d = -24}$

AB Calculus Ch. 3.6 Select HW Problems

Analyzing the Graph of a Function In Exercises 5–24, analyze and sketch a graph of the function. Label any intercepts, relative extrema, points of inflection, and asymptotes. Use a graphing utility to verify your results.

15. $y = x\sqrt{4 - x}$

23. $y = x^5 - 5x$

AB Calculus Ch. 3.4 Select HW Problems

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21. $f(x) = x\sqrt{x + 3}$

23. $f(x) = \dfrac{4}{x^2 + 1}$

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33. $f(x) = x^3 - 3x^2 + 3$	35. $f(x) = x^4 - 4x^3 + 2$
37. $f(x) = x^{2/3} - 3$	39. $f(x) = x + \dfrac{4}{x}$

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Relative minimum: $(5, 1)$

Inflection point: $(4, 2)$

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