4.5b U-Substitution (Definite Integrals)

- Convert bounds to be in terms of u.

Ex. 1 $\int_1^2 2x(x^2-2)^3 dx$

$u = x^2 - 2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$\int \cancel{2x} \cdot u^3 \cdot \frac{du}{\cancel{2x}}$

$\int u^3 du$

convert bounds:

if $x=1$, $u = 1^2 - 2 = -1$

if $x=2$, $u = 2^2 - 2 = 2$

$\int_{-1}^{2} u^3 du$

$\left.\frac{u^4}{4}\right]_{-1}^{2} = \frac{2^4}{4} - \left(\frac{(-1)^4}{4}\right)$

$= \frac{16}{4} - \frac{1}{4} = \boxed{\frac{15}{4}}$

* Do not change bounds or variables back in terms of x.

Ex. 2 $\int_1^5 \frac{x}{\sqrt{2x-1}} dx$

$u = 2x - 1$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$\int \frac{x}{u^{1/2}} \cdot \frac{du}{2}$

* We need to use change of variable method since x does not cancel out.

$u = 2x - 1$

$\frac{u+1}{2} = x$

$= \int \frac{\frac{u+1}{2}}{u^{1/2}} \cdot \frac{du}{2}$

$= \frac{1}{4}\int u^{-1/2}(u+1) du$

$= \frac{1}{4}\int u^{1/2} + u^{-1/2} du$

convert bounds:

if $x=1$, $u = 2(1) - 1 = 1$

if $x=5$, $u = 2(5) - 1 = 9$

$= \frac{1}{4}\frac{u^{3/2}}{3/2} + \frac{1}{4}\frac{u^{1/2}}{1/2}$

$= \left.\frac{1}{6}u^{3/2} + \frac{1}{2}u^{1/2}\right]_1^9$

$= \frac{1}{6}(9)^{3/2} + \frac{1}{2}(9)^{1/2} - \left(\frac{1}{6}(1)^{3/2} + \frac{1}{2}(1)^{1/2}\right)$

$= \frac{1}{6}(27) + \frac{1}{2}(3) - \frac{1}{6} - \frac{1}{2}$

$= \frac{9}{2} + \frac{3}{2} - \frac{1}{6} - \frac{1}{2}$

$= \boxed{\frac{16}{3}}$

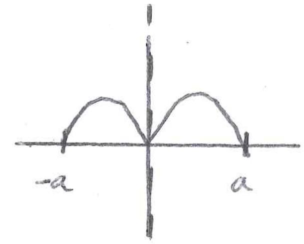
4.5b (continued) Integrals of odd and even functions

Reminder: Suppose $\int_{10}^{3} f(x)dx = 9$, $\int_{-1}^{3} f(x)dx = 5$, Find $\int_{-1}^{10} f(x)dx$

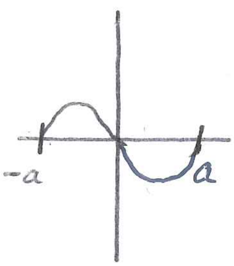
$$\int_{-1}^{10} f(x)dx = \int_{-1}^{3} f(x)dx + \int_{3}^{10} f(x)dx = 5 + (-9) = \boxed{-4}$$

Rules:

Even: $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

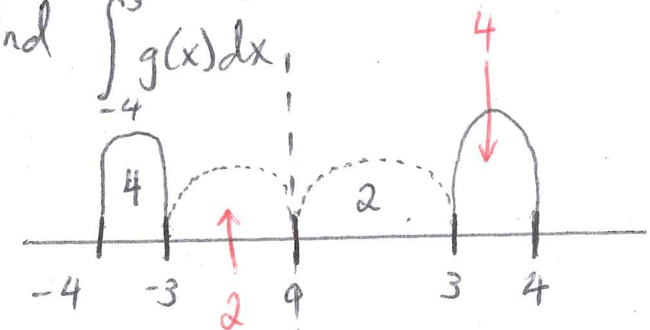


Odd: $\int_{-a}^{a} f(x)dx = 0$



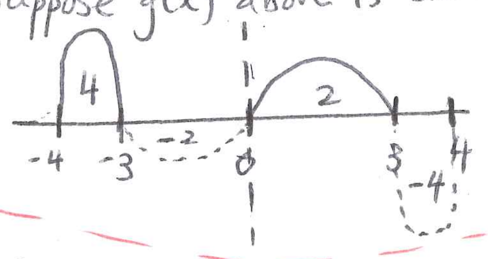
Ex.3 Suppose $g(x)$ is an even function where $\int_{0}^{3} g(x)dx = 2$ and $\int_{-4}^{-3} g(x)dx = 4$

Find $\int_{-4}^{3} g(x)dx$.

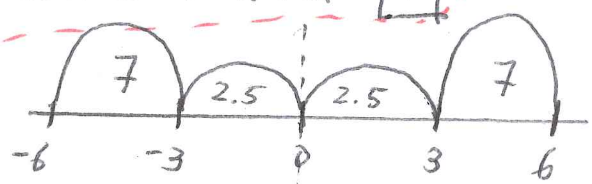


$$\int_{-4}^{3} g(x)dx = 4+2+2 = \boxed{8}$$

Ex.4 Suppose $g(x)$ above is an odd function. Find $\int_{-4}^{3} g(x)dx$



$$\int_{-4}^{3} g(x)dx = 4-2+2 = \boxed{4}$$



Ex.5 $f(x)$ is even

$\int_{3}^{6} f(x)dx = 7$, $\int_{-6}^{3} f(x)dx = 12$. Find $\int_{0}^{6} f(x)dx = 2.5 + 7 = \boxed{9.5}$

$\boxed{4.5b}$ U-substitution (definite integrals) p.305-306 #71-81 odd, 105, 106

73) $\int_1^2 2x^2\sqrt{x^3+1}\,dx$

$= \int_1^2 2x^2(x^3+1)^{1/2}dx$

$u = x^3+1$ | $dx = \frac{du}{3x^2}$

$\frac{du}{dx} = 3x^2$

$\int 2\cancel{x^2}\cdot u^{1/2}\cdot\frac{du}{3\cancel{x^2}}$

$\frac{2}{3}\int u^{1/2}du$

$\frac{2}{3}\left(\frac{u^{3/2}}{3/2}\right)$

$\frac{2}{3}\left(\frac{2}{3}\right)u^{3/2} = \frac{4}{9}u^{3/2}$

$u = x^3+1$

for $x=1$, $u=1+1=2$

for $x=2$, $u=2^3+1=9$

$\left.\frac{4}{9}u^{3/2}\right]_2^9$

$\frac{4}{9}(9)^{3/2} - \frac{4}{9}(2)^{3/2}$

$= \frac{4}{9}(3)^3 - \frac{4}{9}(\sqrt{8})$

$= \frac{4}{\cancel{9}}(\cancel{27}^{\,3}) - \frac{4}{9}(2\sqrt{2})$

$= \boxed{12 - \frac{8\sqrt{2}}{9}}$

77) $\int_1^9 \frac{1}{\sqrt{x}(1+\sqrt{x})^2}dx$

$u = 1+\sqrt{x}$

$u = 1+x^{1/2}$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$dx = 2\sqrt{x}\,du$

$\int \frac{1}{\cancel{\sqrt{x}}\cdot u^2}\cdot 2\cancel{\sqrt{x}}\,du$

$\int \frac{2}{u^2}du$

$\int 2u^{-2}du$

$= \frac{2u^{-1}}{-1} = \frac{-2}{u}$

for $x=1$, $u=1+\sqrt{1}=2$

for $x=9$, $u=1+\sqrt{9}=4$

$\left.\frac{-2}{u}\right]_2^4$

$= \frac{-2}{4} - \left(\frac{-2}{2}\right)$

$= -\frac{1}{2}+1$

$= \boxed{\frac{1}{2}}$

79) $\int_1^2 (x-1)\sqrt{2-x}\,dx$

$\int (x-1)(2-x)^{1/2}dx$

$u = 2-x$

$\frac{du}{dx} = -1$

$dx = -du$

$\int (x-1)u^{1/2}(-du)$

$u = 2-x$

$x = 2-u$

$\int (2-u-1)u^{1/2}(-du)$

$-\int (1-u)u^{1/2}du$

$\int -u^{1/2}+u^{3/2}du$

$-\frac{u^{3/2}}{3/2} + \frac{u^{5/2}}{5/2}$

$-\frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2}$

$u = 2-x$

for $x=1$, $u=2-1=1$

for $x=2$, $u=2-2=0$

$\left.-\frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2}\right]_1^0$

$= -\frac{2}{3}(0) + \frac{2}{5}(0) - \left(-\frac{2}{3}(1)^{3/2} + \frac{2}{5}(1)^{5/2}\right)$

$= 0 + \frac{2}{3} - \frac{2}{5}$

$= \frac{10}{15} - \frac{6}{15}$

$= \boxed{\frac{4}{15}}$

$\boxed{4.5b}$ (continued)

81) $\displaystyle\int_0^{\pi/2} \cos\left(\frac{2x}{3}\right)dx$

$u = \frac{2}{3}x \qquad dx = \frac{3du}{2}$

$\frac{du}{dx} = \frac{2}{3}$

$\int \cos u \left(\frac{3}{2}du\right)$

$\frac{3}{2}\int \cos u\, du$

$= \frac{3}{2}\sin u$

$u = \frac{2}{3}x$

for $x = 0$, $u = \frac{2}{3}(0) = 0$

for $x = \pi/2$, $u = \frac{2}{3}\left(\frac{\pi}{2}\right) = \pi/3$

$\frac{3}{2}\sin u \Big]_0^{\pi/3}$

$= \frac{3}{2}\sin(\pi/3) - \frac{3}{2}\sin(0)$

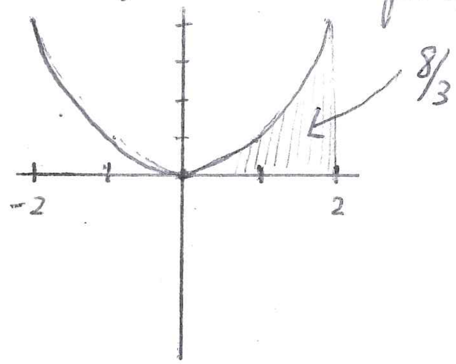
$= \frac{3}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{3}{2}(0)$

$= \boxed{\frac{3\sqrt{3}}{4}}$

105) Use $\displaystyle\int_0^2 x^2 dx = \frac{8}{3}$ to evaluate integral. (Note: $f(x) = x^2$ is an even function)

a) $\displaystyle\int_{-2}^{0} x^2 dx = \boxed{\frac{8}{3}}$

b) $\displaystyle\int_{-2}^{2} x^2 dx = \frac{8}{3} + \frac{8}{3} = \boxed{\frac{16}{3}}$



c) $\displaystyle\int_0^2 -x^2 dx = -\int_0^2 x^2 dx = -\left(\frac{8}{3}\right) = \boxed{\frac{-8}{3}}$

d) $\displaystyle\int_{-2}^{0} 3x^2 dx = 3\int_{-2}^{0} x^2 dx = \cancel{3}\left(\frac{8}{\cancel{3}}\right) = \boxed{8}$

106) Use symmetry of sine and cosine functions to evaluate definite integral

*Recall: $y = \sin x$ is an odd function (origin symmetry)
$y = \cos x$ is an even function (y-axis symmetry)

a) $\displaystyle\int_{-\pi/4}^{\pi/4} \sin x\, dx = \boxed{0}$

b) $\displaystyle\int_{-\pi/4}^{\pi/4} \cos x\, dx = 2\left[\int_0^{\pi/4} \cos x\, dx\right]$ $= \sin x \Big]_0^{\pi/4} = \sin(\pi/4) - \sin(0) = \frac{\sqrt{2}}{2}$

$= 2\left(\frac{\sqrt{2}}{2}\right) = \boxed{\sqrt{2}}$

c) $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x\, dx = 2\left[\int_0^{\pi/2} \cos x\, dx\right]$

$= \sin x \Big]_0^{\pi/2} = \sin(\pi/2) - \sin(0)$

$= 1$

$= 2 \cdot 1 = \boxed{2}$

d) $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x \cos x\, dx$

$\downarrow$

$\boxed{0}$

*Test whether $f(x) = \sin x \cos x$ is an even or odd function:

$f(-x) = \sin(-x)\cos(-x)$

$= -\sin x \cdot \cos x = -\sin x \cos x$

Since $f(-x) = -\sin x \cos x$ $(f(-x) = -f(x))$ this is an odd function (Symmetry about the origin)