5.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Evaluating a Logarithmic Expression In Exercises 1–4, evaluate the expression without using a calculator.

1. $\log_2 \frac{1}{8}$
2. $\log_{27} 9$
3. $\log_7 1$
4. $\log_a \dfrac{1}{a}$

Exponential and Logarithmic Forms of Equations In Exercises 5–8, write the exponential equation as a logarithmic equation or vice versa.

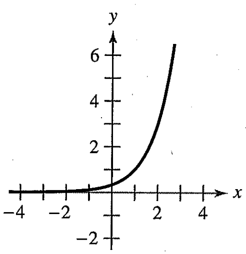
5. (a) $2^3 = 8$
   (b) $3^{-1} = \frac{1}{3}$
6. (a) $27^{2/3} = 9$
   (b) $16^{3/4} = 8$
7. (a) $\log_{10} 0.01 = -2$
   (b) $\log_{0.5} 8 = -3$
8. (a) $\log_3 \frac{1}{9} = -2$
   (b) $49^{1/2} = 7$

Sketching a Graph In Exercises 9–14, sketch the graph of the function by hand.

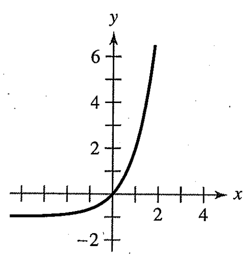
9. $y = 2^x$
10. $y = 4^{x-1}$
11. $y = \left(\frac{1}{3}\right)^x$
12. $y = 2^{x^2}$
13. $h(x) = 5^{x-2}$
14. $y = 3^{-|x|}$

Matching In Exercises 15–18, match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]

(a)



(b)



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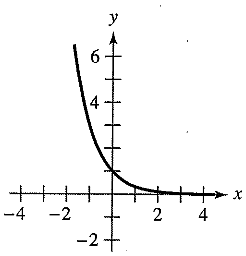
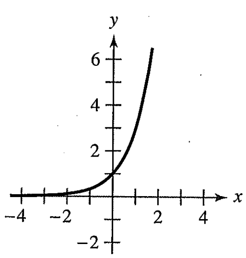


(d)



15. $f(x) = 3^x$
16. $f(x) = 3^{-x}$
17. $f(x) = 3^x - 1$
18. $f(x) = 3^{x-1}$

Solving an Equation In Exercises 19–24, solve for x or b.

19. (a) $\log_{10} 1000 = x$
    (b) $\log_{10} 0.1 = x$
20. (a) $\log_3 \frac{1}{81} = x$
    (b) $\log_6 36 = x$
21. (a) $\log_3 x = -1$
    (b) $\log_2 x = -4$
22. (a) $\log_b 27 = 3$
    (b) $\log_b 125 = 3$
23. (a) $x^2 - x = \log_5 25$
    (b) $3x + 5 = \log_2 64$
24. (a) $\log_3 x + \log_3(x - 2) = 1$
    (b) $\log_{10}(x + 3) - \log_{10} x = 1$

Solving an Equation In Exercises 25–34, solve the equation accurate to three decimal places.

25. $3^{2x} = 75$
26. $5^{6x} = 8320$
27. $2^{3-z} = 625$
28. $3(5^{x-1}) = 86$
29. $\left(1 + \dfrac{0.09}{12}\right)^{12t} = 3$
30. $\left(1 + \dfrac{0.10}{365}\right)^{365t} = 2$
31. $\log_2(x - 1) = 5$
32. $\log_{10}(t - 3) = 2.6$
33. $\log_3 x^2 = 4.5$
34. $\log_5 \sqrt{x - 4} = 3.2$

Verifying Inverse Functions In Exercises 35 and 36, illustrate that the functions are inverse functions of each other by sketching their graphs on the same set of coordinate axes.

35. $f(x) = 4^x$
    $g(x) = \log_4 x$
36. $f(x) = 3^x$
    $g(x) = \log_3 x$

Finding a Derivative In Exercises 37–58, find the derivative of the function. (*Hint:* In some exercises, you may find it helpful to apply logarithmic properties *before* differentiating.)

37. $f(x) = 4^x$
38. $f(x) = 3^{4x}$
39. $y = 5^{-4x}$
40. $y = 6^{3x-4}$
41. $f(x) = x\,9^x$
42. $y = x(6^{-2x})$
43. $g(t) = t^2 2^t$
44. $f(t) = \dfrac{3^{2t}}{t}$
45. $h(\theta) = 2^{-\theta} \cos \pi\theta$
46. $g(\alpha) = 5^{-\alpha/2} \sin 2\alpha$
47. $y = \log_4(5x + 1)$
48. $y = \log_3(x^2 - 3x)$
49. $h(t) = \log_5(4 - t)^2$
50. $g(t) = \log_2(t^2 + 7)^3$
51. $y = \log_5 \sqrt{x^2 - 1}$
52. $f(x) = \log_2 \sqrt[3]{2x + 1}$
53. $f(x) = \log_2 \dfrac{x^2}{x - 1}$
54. $y = \log_{10} \dfrac{x^2 - 1}{x}$
55. $h(x) = \log_3 \dfrac{x\sqrt{x - 1}}{2}$
56. $g(x) = \log_5 \dfrac{4}{x^2\sqrt{1 - x}}$
57. $g(t) = \dfrac{10 \log_4 t}{t}$
58. $f(t) = t^{3/2} \log_2 \sqrt{t + 1}$

Finding an Equation of a Tangent Line In Exercises 59–62, find an equation of the tangent line to the graph of the function at the given point.

59. $y = 2^{-x}$, $(-1, 2)$
60. $y = 5^{x-2}$, $(2, 1)$
61. $y = \log_3 x$, $(27, 3)$
62. $y = \log_{10} 2x$, $(5, 1)$

Logarithmic Differentiation In Exercises 63–66, use logarithmic differentiation to find dy/dx.

63. $y = x^{2/x}$
64. $y = x^{x-1}$
65. $y = (x - 2)^{x+1}$
66. $y = (1 + x)^{1/x}$

Finding an Equation of a Tangent Line **In Exercises 67–70, find an equation of the tangent line to the graph of the function at the given point.**

67. $y = x^{\sin x}, \quad \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

68. $y = (\sin x)^{2x}, \quad \left(\frac{\pi}{2}, 1\right)$

69. $y = (\ln x)^{\cos x}, \quad (e, 1)$

70. $y = x^{1/x}, \quad (1, 1)$

Finding an Indefinite Integral **In Exercises 71–78, find the indefinite integral.**

71. $\int 3^x \, dx$

72. $\int 8^{-x} \, dx$

73. $\int (x^2 + 2^{-x}) \, dx$

74. $\int (x^4 + 5^x) \, dx$

75. $\int x(5^{-x^2}) \, dx$

76. $\int (x + 4)6^{(x+4)^2} \, dx$

77. $\int \frac{3^{2x}}{1 + 3^{2x}} \, dx$

78. $\int 2^{\sin x} \cos x \, dx$

Evaluating a Definite Integral **In Exercises 79–82, evaluate the definite integral.**

79. $\int_{-1}^{2} 2^x \, dx$

80. $\int_{-4}^{4} 3^{x/4} \, dx$

81. $\int_{0}^{1} (5^x - 3^x) \, dx$

82. $\int_{1}^{3} (7^x - 4^x) \, dx$

Area **In Exercises 83 and 84, find the area of the region bounded by the graphs of the equations.**

83. $y = 3^x, y = 0, x = 0, x = 3$

84. $y = 3^{\cos x} \sin x, y = 0, x = 0, x = \pi$

WRITING ABOUT CONCEPTS

85. **Analyzing a Logarithmic Equation** Consider the function $f(x) = \log_{10} x$.
   (a) What is the domain of f?
   (b) Find f^{-1}.
   (c) Let x be a real number between 1000 and 10,000. Determine the interval in which $f(x)$ will be found.
   (d) Determine the interval in which x will be found if $f(x)$ is negative.
   (e) When $f(x)$ is increased by one unit, x must have been increased by what factor?
   (f) Find the ratio of x_1 to x_2 given that $f(x_1) = 3n$ and $f(x_2) = n$.

86. **Comparing Rates of Growth** Order the functions

$$f(x) = \log_2 x, \quad g(x) = x^x, \quad h(x) = x^2, \quad \text{and} \quad k(x) = 2^x$$

from the one with the greatest rate of growth to the one with the least rate of growth for large values of x.

87. **Inflation** When the annual rate of inflation averages 5% over the next 10 years, the approximate cost C of goods or services during any year in that decade is

$$C(t) = P(1.05)^t$$

where t is the time in years and P is the present cost.
   (a) The price of an oil change for your car is presently \$24.95. Estimate the price 10 years from now.
   (b) Find the rates of change of C with respect to t when $t = 1$ and $t = 8$.
   (c) Verify that the rate of change of C is proportional to C. What is the constant of proportionality?

88. **Depreciation** After t years, the value of a car purchased for \$25,000 is

$$V(t) = 25{,}000\left(\tfrac{3}{4}\right)^t.$$

   (a) Use a graphing utility to graph the function and determine the value of the car 2 years after it was purchased.
   (b) Find the rates of change of V with respect to t when $t = 1$ and $t = 4$.
   (c) Use a graphing utility to graph $V'(t)$ and determine the horizontal asymptote of $V'(t)$. Interpret its meaning in the context of the problem.

Compound Interest **In Exercises 89–92, complete the table by determining the balance A for P dollars invested at rate r for t years and compounded n times per year.**

n	1	2	4	12	365	Continuous Compounding
A						

89. $P = \$1000$
   $r = 3\frac{1}{2}\%$
   $t = 10$ years

90. $P = \$2500$
   $r = 6\%$
   $t = 20$ years

91. $P = \$1000$
   $r = 5\%$
   $t = 30$ years

92. $P = \$4000$
   $r = 4\%$
   $t = 15$ years

Compound Interest **In Exercises 93–96, complete the table by determining the amount of money P (present value) that should be invested at rate r to produce a balance of \$100,000 in t years.**

t	1	10	20	30	40	50
P						

93. $r = 5\%$
   Compounded continuously

94. $r = 3\%$
   Compounded continuously

95. $r = 5\%$
   Compounded monthly

96. $r = 2\%$
   Compounded daily

97. Compound Interest Assume that you can earn 6% on an investment, compounded daily. Which of the following options would yield the greatest balance after 8 years?

(a) \$20,000 now (b) \$30,000 after 8 years

(c) \$8000 now and \$20,000 after 4 years

(d) \$9000 now, \$9000 after 4 years, and \$9000 after 8 years

98. Compound Interest Consider a deposit of \$100 placed in an account for 20 years at $r\%$ compounded continuously. Use a graphing utility to graph the exponential functions describing the growth of the investment over the 20 years for the following interest rates. Compare the ending balances for the three rates.

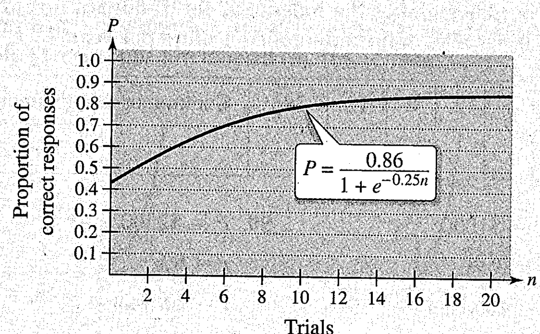
(a) $r = 3\%$ (b) $r = 5\%$ (c) $r = 6\%$

99. Timber Yield The yield V (in millions of cubic feet per acre) for a stand of timber at age t is $V = 6.7e^{(-48.1)/t}$, where t is measured in years.

(a) Find the limiting volume of wood per acre as t approaches infinity.

(b) Find the rates at which the yield is changing when $t = 20$ years and $t = 60$ years.

100. HOW DO YOU SEE IT? The graph shows the proportion P of correct responses after n trials in a group project in learning theory.



(a) What is the limiting proportion of correct responses as n approaches infinity?

(b) What happens to the rate of change of the proportion in the long run?

101. Population Growth A lake is stocked with 500 fish, and the population increases according to the logistic curve

$$p(t) = \frac{10,000}{1 + 19e^{-t/5}}$$

where t is measured in months.

(a) Use a graphing utility to graph the function.

(b) What is the limiting size of the fish population?

(c) At what rates is the fish population changing at the end of 1 month and at the end of 10 months?

(d) After how many months is the population increasing most rapidly?

102. Modeling Data The breaking strengths B (in tons) of steel cables of various diameters d (in inches) are shown in the table.

d	0.50	0.75	1.00	1.25	1.50	1.75
B	9.85	21.8	38.3	59.2	84.4	114.0

(a) Use the regression capabilities of a graphing utility to fit an exponential model to the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Find the rates of growth of the model when $d = 0.8$ and $d = 1.5$.

103. Comparing Models The numbers y of pancreas transplants in the United States for the years 2004 through 2010 are shown in the table, with $x = 4$ corresponding to 2004. (*Source: Organ Procurement and Transplantation Network*)

x	4	5	6	7	8	9	10
y	603	542	466	468	436	376	350

(a) Use the regression capabilities of a graphing utility to find the following models for the data.

$y_1 = ax + b$ $\quad y_2 = a + b \ln x$

$y_3 = ab^x$ $\quad y_4 = ax^b$

(b) Use a graphing utility to plot the data and graph each of the models. Which model do you think best fits the data?

(c) Interpret the slope of the linear model in the context of the problem.

(d) Find the rate of change of each of the models for the year 2008. Which model is decreasing at the greatest rate in 2008?

104. An Approximation of *e* Complete the table to demonstrate that e can also be defined as

$$\lim_{x \to 0^+} (1 + x)^{1/x}.$$

x	1	10^{-1}	10^{-2}	10^{-4}	10^{-6}
$(1 + x)^{1/x}$					

Modeling Data In Exercises 105 and 106, find an exponential function that fits the experimental data collected over time *t*.

105.

t	0	1	2	3	4
y	1200.00	720.00	432.00	259.20	155.52

106.

t	0	1	2	3	4
y	600.00	630.00	661.50	694.58	729.30

Using Properties of Exponents In Exercises 107–110, find the exact value of the expression.

107. $5^{1/\ln 5}$ **108.** $6^{\ln 10/\ln 6}$

109. $9^{1/\ln 3}$ **110.** $32^{1/\ln 2}$

True or False? In Exercises 111–116, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

111. $e = \dfrac{271{,}801}{99{,}900}$

112. If $f(x) = \ln x$, then $f(e^{n+1}) - f(e^n) = 1$ for any value of n.

113. The functions $f(x) = 2 + e^x$ and $g(x) = \ln(x - 2)$ are inverse functions of each other.

114. The exponential function $y = Ce^x$ is a solution of the differential equation

$$\frac{d^n y}{dx^n} = y, \quad n = 1, 2, 3, \ldots .$$

115. The graphs of $f(x) = e^x$ and $g(x) = e^{-x}$ meet at right angles.

116. If $f(x) = g(x)e^x$, then the only zeros of f are the zeros of g.

117. Comparing Functions

(a) Show that $(2^3)^2 \neq 2^{(3^2)}$.

(b) Are $f(x) = (x^x)^x$ and $g(x) = x^{(x^x)}$ the same function? Why or why not?

(c) Find $f'(x)$ and $g'(x)$.

118. Finding an Inverse Function Let

$$f(x) = \frac{a^x - 1}{a^x + 1}$$

for $a > 0$, $a \neq 1$. Show that f has an inverse function. Then find f^{-1}.

119. Logistic Differential Equation Show that solving the logistic differential equation

$$\frac{dy}{dt} = \frac{8}{25}y\left(\frac{5}{4} - y\right), \quad y(0) = 1$$

results in the logistic growth function in Example 7.

$$\left[\textit{Hint: } \frac{1}{y\left(\frac{5}{4} - y\right)} = \frac{4}{5}\left(\frac{1}{y} + \frac{1}{\frac{5}{4} - y}\right)\right]$$

120. Using Properties of Exponents Given the exponential function $f(x) = a^x$, show that

(a) $f(u + v) = f(u) \cdot f(v)$.

(b) $f(2x) = [f(x)]^2$.

121. Tangent Lines

(a) Determine y' given $y^x = x^y$.

(b) Find the slope of the tangent line to the graph of $y^x = x^y$ at each of the following points.

(i) (c, c) (ii) $(2, 4)$ (iii) $(4, 2)$

(c) At what points on the graph of $y^x = x^y$ does the tangent line not exist?

PUTNAM EXAM CHALLENGE

122. Which is greater

$$\left(\sqrt{n}\right)^{\sqrt{n+1}} \quad \text{or} \quad \left(\sqrt{n+1}\right)^{\sqrt{n}}$$

where $n > 8$?

123. Show that if x is positive, then

$$\log_e\left(1 + \frac{1}{x}\right) > \frac{1}{1 + x}.$$

SECTION PROJECT

Using Graphing Utilities to Estimate Slope

Let $f(x) = \begin{cases} |x|^x, & x \neq 0 \\ 1, & x = 0. \end{cases}$

(a) Use a graphing utility to graph f in the viewing window $-3 \leq x \leq 3$, $-2 \leq y \leq 2$. What is the domain of f?

(b) Use the *zoom* and *trace* features of a graphing utility to estimate

$$\lim_{x \to 0} f(x).$$

(c) Write a short paragraph explaining why the function f is continuous for all real numbers.

(d) Visually estimate the slope of f at the point $(0, 1)$.

(e) Explain why the derivative of a function can be approximated by the formula

$$\frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$

for small values of Δx. Use this formula to approximate the slope of f at the point $(0, 1)$.

$$f'(0) \approx \frac{f(0 + \Delta x) - f(0 - \Delta x)}{2\Delta x} = \frac{f(\Delta x) - f(-\Delta x)}{2\Delta x}$$

What do you think the slope of the graph of f is at $(0, 1)$?

(f) Find a formula for the derivative of f and determine $f'(0)$. Write a short paragraph explaining how a graphing utility might lead you to approximate the slope of a graph incorrectly.

(g) Use your formula for the derivative of f to find the relative extrema of f. Verify your answer using a graphing utility.

FOR FURTHER INFORMATION For more information on using graphing utilities to estimate slope, see the article "Computer-Aided Delusions" by Richard L. Hall in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.