## 6.3 Notes: Differential Equations and Separation of Variables

Separation of Variables: Rearrange equation with y and dy (dependent variable) on the left and the x, dx (independent variable) on the right side of the equation

1) Solve the differential equation $\frac{dy}{dx} = \frac{2x}{y}$

2) Solve $\frac{dy}{dx} = x(1+y)$

3) Find a general solution of $2x + 3yy' = 0$. Then find the particular solution, $y = f(x)$, if the solution passes through the point $(1, -2)$.

2

4) Find a general solution to $yy' = 6\cos(\pi x)$. Then find the particular solution, $y = f(x)$, if the function passes through the point (1, 2).

5) Solve $y' = (x + 1)y$

3

# Solving Differential Equations: Summary of Steps

## I. Separation of variables

1. Rewrite $y'$ as $\frac{dy}{dx}$ $(or\ \frac{dy}{dt}\ if\ time\ t\ is\ independent\ variable)$

   ***numerator** of the differential fraction (dy) is the **dependent variable**.

   * Denominator of the differential fraction(dx) is the independent variable.

2. Group the dependent variables together on **left side** of equation $(y\ \&\ dy)$

3. Independent variables on the **right side** of equation $(x\ \&\ dx\quad or\quad t\ \&\ dt)$

4. Start by cross-multiplying equation to rearrange all variables in the numerator.

5. Divide terms and/or variables to the other side if the variables are not yet separated.

6. Remember that $\boldsymbol{dy}$ & $\boldsymbol{dx}$ terms need to be in the **numerator location** of their respective sides, never in the denominator once variables are separated.

7. It's ok to have variable(s) $\boldsymbol{x\ or\ y}$ in the denominator after all variables are separated.

8. If parentheses are presented in the differential equation, keep them. Don't expand or distribute. The parentheses are there to help you group the terms that need to stay together.

9. Try to keep the left side of the equation with just the bare minimum terms if possible. $(\boldsymbol{y}, \boldsymbol{dy}\ \&\ any\ terms\ grouped\ with\ y\ in\ parentheses)$. Any coefficient constants keep (or move) to the right side of the equation.

## II. Antidifferentiation (Take Integral of both sides)

1. Treat each side as a separate problem and take the appropriate indefinite integral of each side. (Power Rule, other Integral rules, or U-Substitution)

2. We only need to display a "+C" on the right side of the equation.

3. Solve for the "+C" constant

   a) Option 1: Solve for the **C** immediately after it appears. Use the ordered pair given in the problem to solve for C. (my preference is solving for +C if y is raised to a power; example: $\boldsymbol{y^2}, \boldsymbol{y^3}, \boldsymbol{y^{3/2}}$)

   b) Option 2: Wait to solve for **C**. First, clean up the equation by solving for y (isolate y variable) on the left side of the equation (my preference is solving for y first if I see $\boldsymbol{ln(y)}$ on the left side of the equation). Finally, solve for the value of **C** using the given ordered pair.

4

Calculus AB Chapter 6.2 Notes: **Solving Differential Equation Word Problems**

1. Direct proportion equation :

2. Inverse (indirect) proportion equation:

3. k is called the ______________________________

**Exponential Growth/Decay class examples**

1) If the rate of change of y varies directly with the value of y, find the general equation:

2) The rate of increase of the population of a certain city is proportional to the population. If the population in 1930 was 50,000 and in 1960 it was 75,000, what was the expected population in 1990?

6

3) The rate of decay of radium is proportional to the amount present at any time. If 60 mg of radium are present now and its half-life is 1690 years, how much radium will be present 100 years from now?

4. In a certain culture where the rate of growth of bacteria is proportional to the amount present, the number triples in 3 hours.
A) If at the end of 12 hours there were 10 million bacteria, how many were present initially?
B) Find the specific exponential growth equation

**<u>Note for homework</u>**: Newton's law of cooling: the rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding medium

**Differential Equations Unit** **Slope Fields (Direction Fields) Notes**

Slope Fields: a graphical approach to look at all the solutions of a differential equation. Slope fields consists of short line segments representing slope (steepness) sketched at lots of different points

These line segments are the tangents to a family of solution curves for the differential equation at various points. The tangents show the direction in which the solution curves will follow. Slope fields are useful in sketching solution curves without having to solve a differential equation algebraically.

Steps:

1) Identify the ordered pairs indicated on the graph.
2) Plug in the ordered pairs in the differential equation to find slope
3) Sketch a short line segment representing the slope through the given point
4) Repeat this for all remaining ordered pairs.

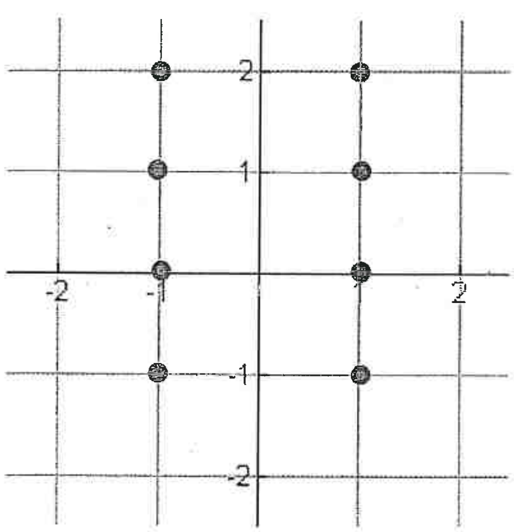
*Use the differential equation to find the individual slope segments, creating the slope field (ex: $\frac{dy}{dx} = x$)

*Use the solution of the differential equation to match with the shape of the slope field: (ex: $y = \frac{1}{2}x^2 + C$)

**Example 1**: Sketch a slope field for the given differential equation at the indicated eight points.

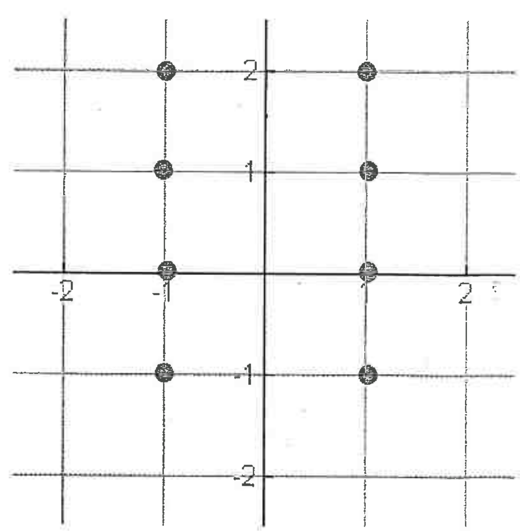
a) $\frac{dy}{dx} = x - 2y$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| x | | | | | | | | |
| y | | | | | | | | |
| y' | | | | | | | | |




b) $\frac{dy}{dx} = \frac{2-x}{y}$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| x | | | | | | | | |
| y | | | | | | | | |
| y' | | | | | | | | |



8

Determine the differential equation being graphed by each of the slope fields below. Then sketch a solution that passes through the indicated point.

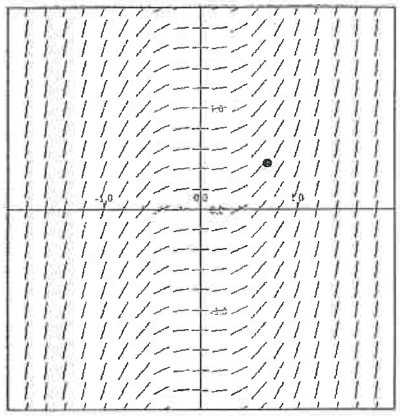
2. a) $\frac{dy}{dx} = x^3$

b) $\frac{dy}{dx} = 3x^2$

c) $\frac{dy}{dx} = 2x + y$

d) $\frac{dy}{dx} = \frac{x}{y}$

e) $\frac{dy}{dx} = \ln x$



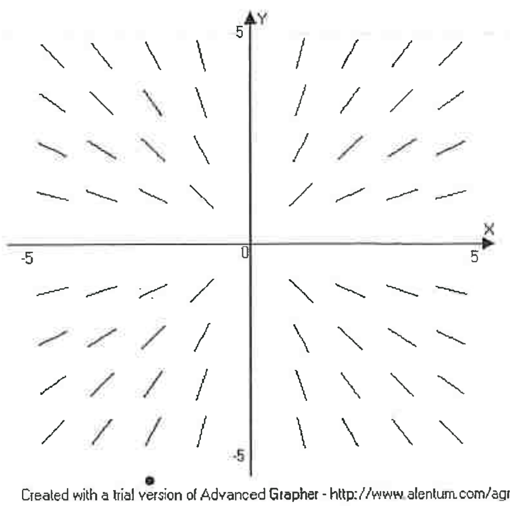
3. a) $\frac{dy}{dx} = x - 2$

b) $\frac{dy}{dx} = x^3$

c) $\frac{dy}{dx} = x - y$

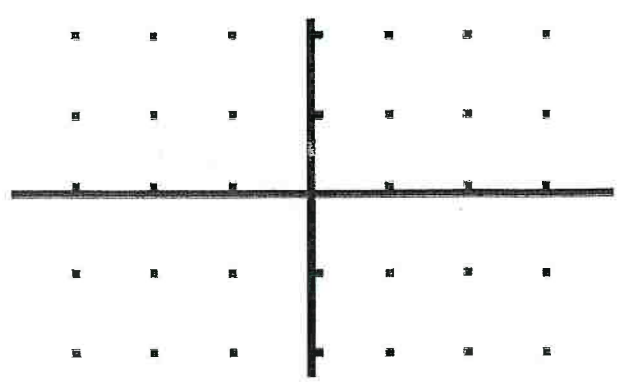
d) $\frac{dy}{dx} = \frac{y}{x}$

e) $\frac{dy}{dx} = e^y$




Sketch slope fields for the following differential equation. Then find the general solution analytically

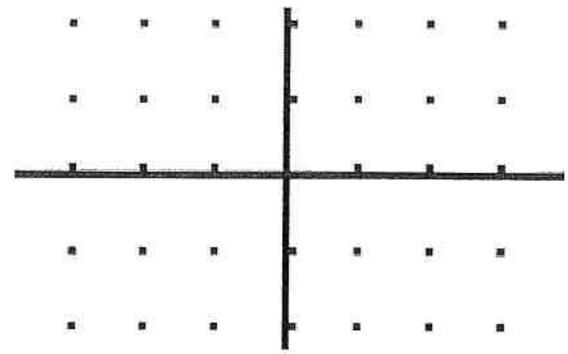
4. $\frac{dy}{dx} = \frac{-x}{y}$



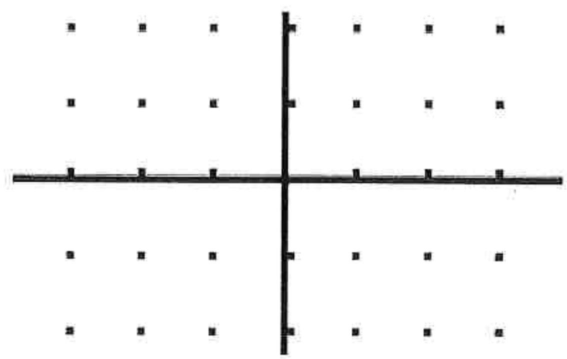
## Slope Fields Homework

Sketch slope fields for each of the following differential equations. Then find the general solution analytically. For problems 4 through 6, find the general solution first, then the specific solution that passes through the given point and state the domain of that solution

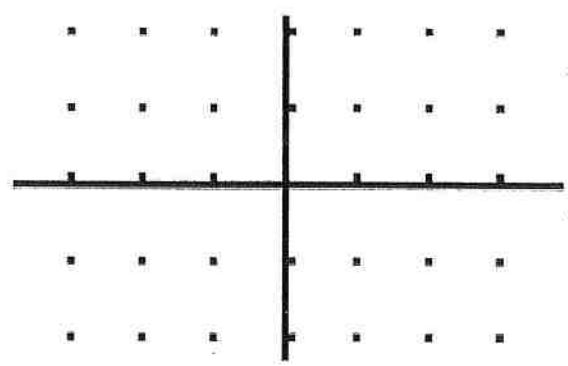
1. $\frac{dy}{dx} = 4 - y$



2. $\frac{dy}{dx} = 3x^2$

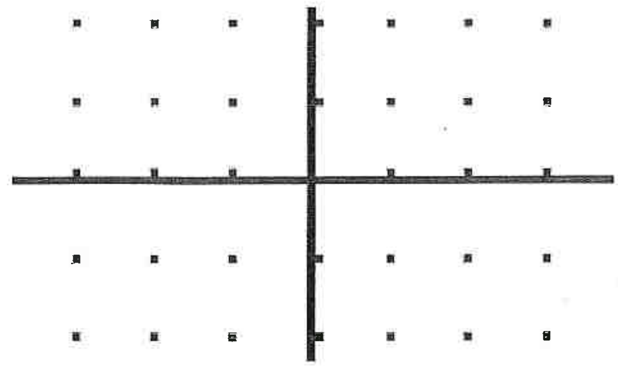


3. $\frac{dy}{dx} = \frac{-x}{y}$

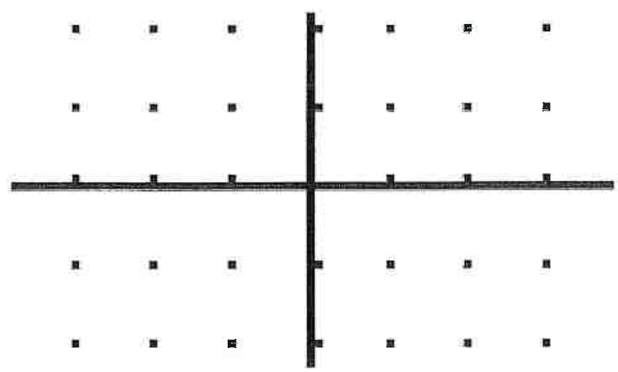


10

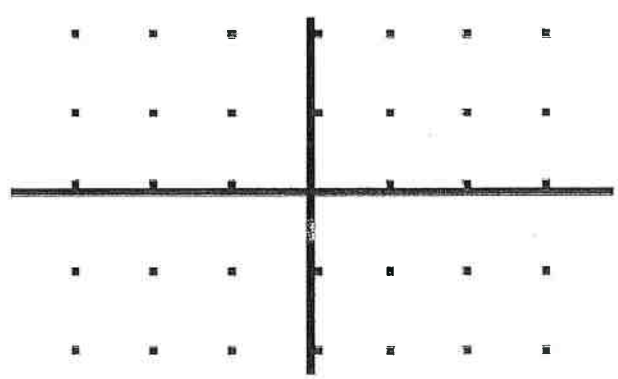
4. $\frac{dy}{dx} = \frac{1}{y}$ point (1, 3)



5. $\frac{dy}{dx} = \frac{xy}{8}$ point (0, -2)



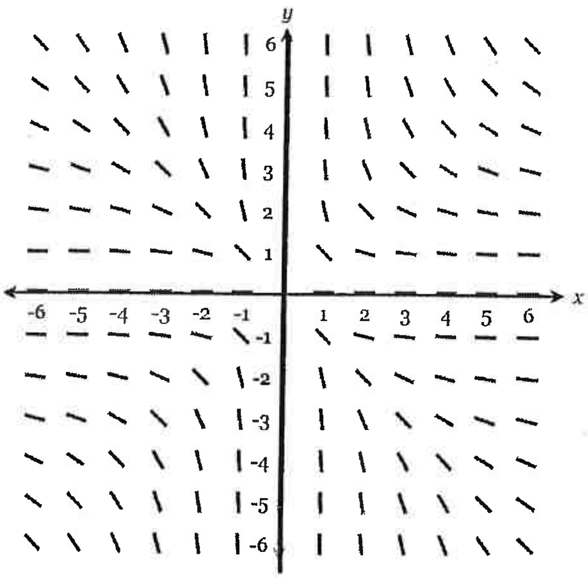
6. $\frac{dy}{dx} = \frac{y}{x}$ point (3, -6)



## Slope Fields Practice WS

Select the differential equation that matches the given slope field.

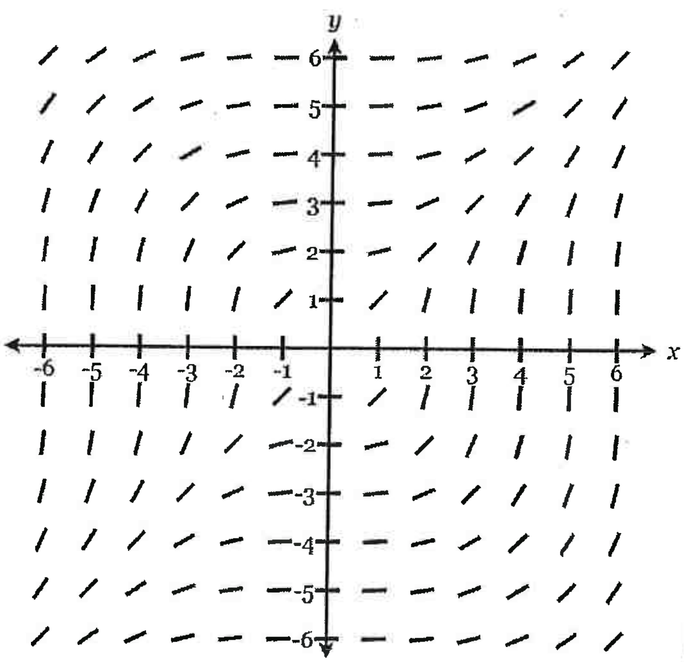
1)




- $\frac{dy}{dx} = -xy^2$
- $\frac{dy}{dx} = -x^2y$
- $\frac{dy}{dx} = \frac{x^2}{y}$
- $\frac{dy}{dx} = -\frac{y^2}{x^2}$

---

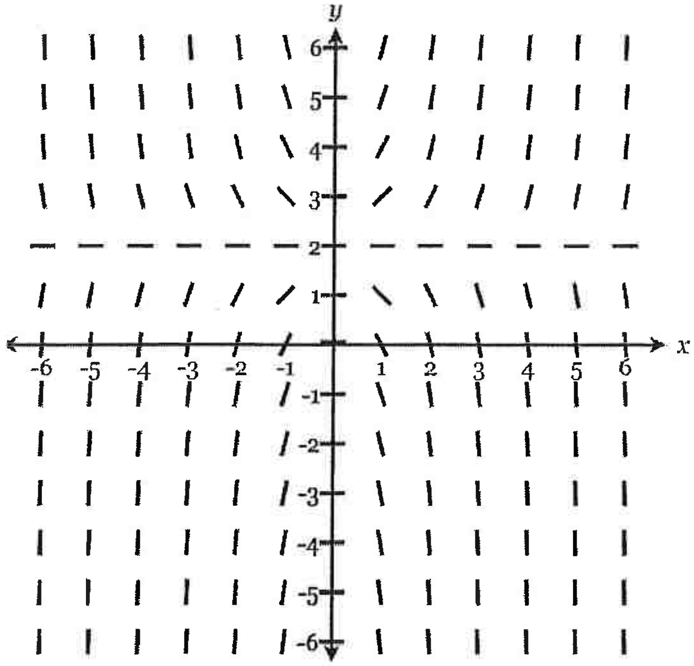
2)




- $\frac{dy}{dx} = -xy^2$
- $\frac{dy}{dx} = \frac{y^2}{x}$
- $\frac{dy}{dx} = \frac{x^2}{y^2}$
- $\frac{dy}{dx} = \frac{-y^2}{x^2}$

12

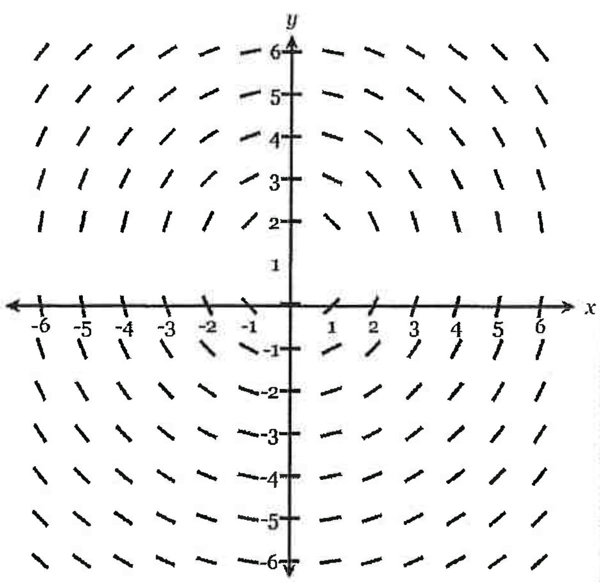
3)




- $\frac{dy}{dx} = \frac{x^2}{(y-2)^2}$
- $\frac{dy}{dx} = -x^2(y-2)^2$
- $\frac{dy}{dx} = x(y-2)$
- $\frac{dy}{dx} = -\frac{y-2}{x}$

---

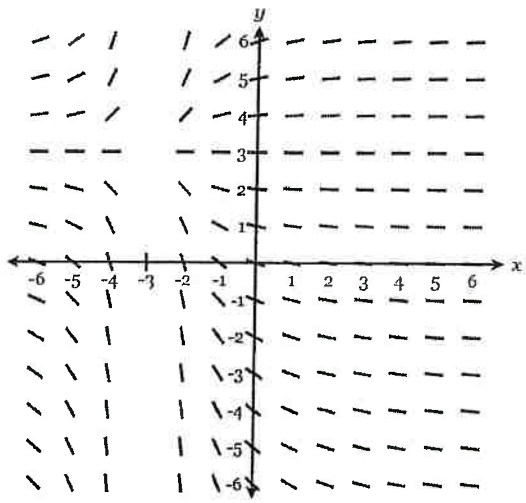
4)




- $\frac{dy}{dx} = -\frac{x}{(y-1)^2}$
- $\frac{dy}{dx} = -\frac{(y-1)^2}{x}$
- $\frac{dy}{dx} = -\frac{x}{y-1}$
- $\frac{dy}{dx} = \frac{x^2}{y-1}$

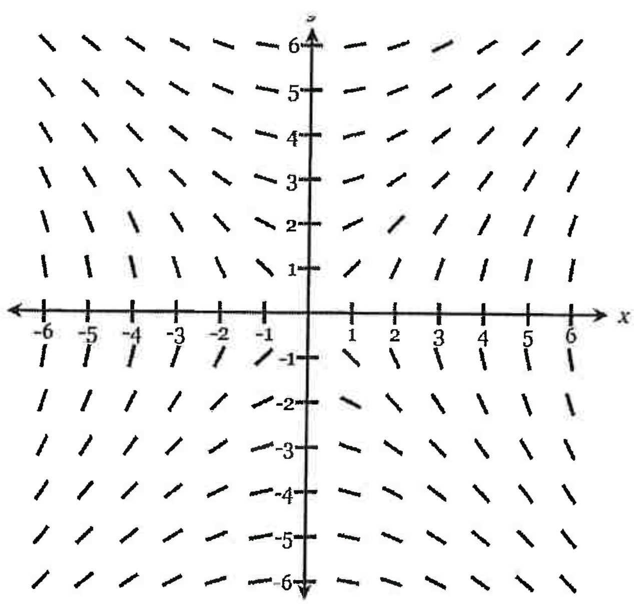
13

5)




- $\frac{dy}{dx} = -(x+3)(y-3)$
- $\frac{dy}{dx} = \frac{(y-3)^2}{(x+3)^2}$
- $\frac{dy}{dx} = -\frac{x+3}{y-3}$
- $\frac{dy}{dx} = \frac{y-3}{(x+3)^2}$

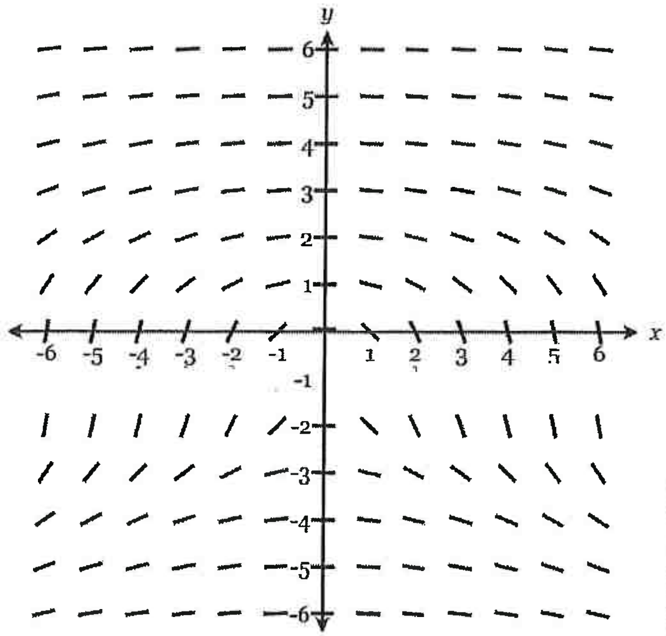
6)




- $\frac{dy}{dx} = \frac{x}{y}$
- $\frac{dy}{dx} = -xy$
- $\frac{dy}{dx} = xy$
- $\frac{dy}{dx} = -x^2y$

14

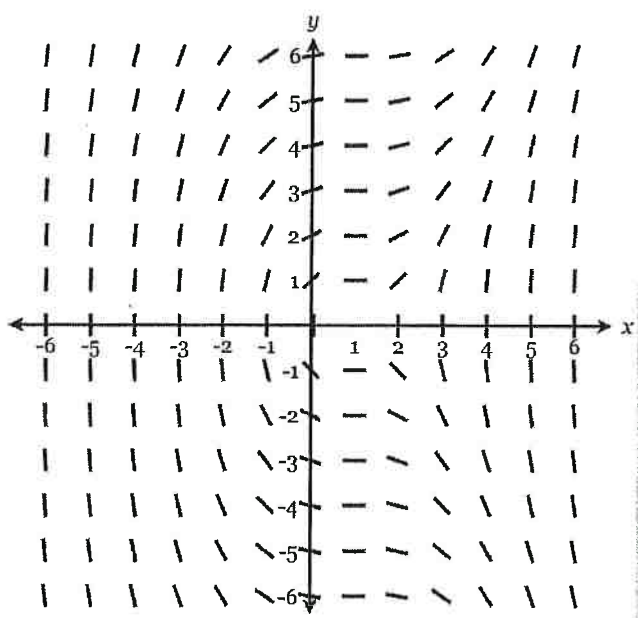
7)



- $\frac{dy}{dx} = -\frac{x}{(y+1)^2}$
- $\frac{dy}{dx} = x^2(y+1)^2$
- $\frac{dy}{dx} = \frac{(y+1)^2}{x}$
- $\frac{dy}{dx} = x^2(y+1)$

---

8)



- $\frac{dy}{dx} = \frac{(x-1)^2}{y}$
- $\frac{dy}{dx} = \frac{y}{x-1}$
- $\frac{dy}{dx} = \frac{x-1}{y^2}$
- $\frac{dy}{dx} = -\frac{y}{(x-1)^2}$

## Solving Differential Equations Task (part 2)

1)

The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time $t$ days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

16

[redacted]

**Solve the below differential equation:**

2) $y' - xycos(x^2) = 0$ given $y(0) = e$ a) Find general solution b) Find particular solution

## Solving Differential Equations: Additional Practice Problems

3) Solve the Differential Equation: $\boldsymbol{y \ln x^4 - xy' = 0}$

18

4) a) Find the general solution b) Find the particular solution

$$yy' - 2e^{3x} = 0 \qquad y(0) = 5$$

**Solving Differential Equations Task (Continued)**

1) The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time $t$ days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

$$\frac{dB}{dt} = \frac{100-B}{5}$$

$$5\,dB = (100-B)\,dt$$

$$\frac{dB}{100-B} = \frac{dt}{5}$$

$$\int \frac{dB}{100-B} = \frac{1}{5}\int dt$$

$u = 100 - B$

$\frac{du}{dB} = -1$

$du = -dB$

$dB = -du$

$$\int \frac{-1\,du}{u}$$

$$= -\ln|u|$$

$$-\ln|100-B| = \frac{1}{5}t + c$$

$$\ln|100-B| = -\frac{1}{5}t + C$$

$$e^{\ln|100-B|} = e^{-\frac{1}{5}t + c}$$

$$|100-B| = e^{-1/5t} \cdot e^{c}$$

$$|100-B| = e^{-\frac{1}{5}t} \cdot C$$

$$|100-B| = Ce^{-\frac{1}{5}t}$$

$$100 - B = Ce^{-\frac{1}{5}t}$$

( $t$ time, $B$ Bird weight)

(0, 20)

$$100 - Ce^{-\frac{1}{5}t} = B$$

$B = 100 - Ce^{-\frac{1}{5}t}$ (general equation)

plug in (0,20)

$$20 = 100 - Ce^{-\frac{1}{5}(0)}$$

$$20 = 100 - C$$

$$C = 80$$

$$\boxed{B = 100 - 80e^{-\frac{1}{5}t}}$$

$$\boxed{B(t) = 100 - 80e^{-\frac{1}{5}t}}$$

2) $y' - xy\cos(x^2) = 0$ $\quad y(0) = e$ Find particular solution

$$\frac{dy}{dx} = xy\cos(x^2)$$

$$dy = xy\cos(x^2)\,dx$$

$$\frac{dy}{y} = x\cos(x^2)\,dx$$

$$\int \frac{1}{y}\,dy = \int x\cos(x^2)\,dx$$

$u = x^2$ $\quad dx = \frac{du}{2x}$

$\frac{du}{dx} = 2x$

$$\int \cancel{x}\cos u \cdot \frac{du}{2\cancel{x}}$$

$$\int \frac{1}{y}\,dy = \frac{1}{2}\int \cos u\,du$$

$$\ln|y| = \frac{1}{2}\sin u + C$$

$$\ln|y| = \frac{1}{2}\sin(x^2) + C$$

$$e^{\ln|y|} = e^{\frac{1}{2}\sin(x^2) + c}$$

$$|y| = e^{\frac{1}{2}\sin(x^2)} \cdot e^{c}$$

$$|y| = Ce^{\frac{1}{2}\sin(x^2)}$$

$$\boxed{y = Ce^{\frac{1}{2}\sin(x^2)}}$$ general solution

$$y = Ce^{\frac{1}{2}\sin(x^2)}$$ plug in $y(0) = e$

$$e = Ce^{\frac{1}{2}\sin(0)^2}$$

$$e = Ce^{0}$$

$$e = C$$

$$y = e \cdot e^{\frac{1}{2}\sin(x^2)}$$

$$\boxed{y = e^{\frac{1}{2}\sin(x^2)+1}}$$

186

18c

**Solving Differential Equations: Additional Practice Problems**

3)Solve the Differential Equation: $y\ln x^4 - xy' = 0$

$$y\ln x^4 - x\left(\frac{dy}{dx}\right) = 0$$

$$\frac{-x\,dy}{dx} = \frac{-y\ln x^4}{1}$$

$$-x\,dy = -y\ln x^4\,dx$$

$$x\,dy = y\ln x^4\,dx$$

$$\frac{dy}{y} = \frac{\ln x^4}{x}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{4\ln x}{x}\,dx$$

$u = \ln x$, $\frac{du}{dx} = \frac{1}{x}$, $dx = x\,du$

$$4\int \frac{u}{x}\cdot x\,du \rightarrow 4\int u\,du$$

$$4\left(\frac{u^2}{2}\right)$$

$$\ln|y| = 2u^2 + C$$

$$\ln|y| = 2(\ln x)^2 + C$$

$$e^{\ln|y|} = e^{2(\ln x)^2 + C}$$

$$|y| = e^{2(\ln x)^2}\cdot e^{C}$$

$$|y| = e^{2(\ln x)^2}\cdot C$$

$$\boxed{y = Ce^{2(\ln x)^2}}$$

4) a) Find the general solution b) Find the particular solution

$yy' - 2e^{3x} = 0 \qquad y(0) = 5$

$$y\left(\frac{dy}{dx}\right) - 2e^{3x} = 0$$

$$\frac{y\,dy}{dx} = \frac{2e^{3x}}{1}$$

$$y\,dy = 2e^{3x}\,dx$$

$$\int y\,dy = \int 2e^{3x}\,dx$$

$u = 3x$, $\frac{du}{dx} = 3$, $dx = \frac{du}{3}$

$$2\int e^{u}\cdot\frac{du}{3}$$

$$\int y\,dy = \frac{2}{3}\int e^{u}\,du$$

$$\frac{y^2}{2} = \frac{2}{3}e^{3x} + C$$

solve for C: $y(0) = 5$

$$\frac{5^2}{2} = \frac{2}{3}e^{3(0)} + C$$

$$\frac{25}{2} = \frac{2}{3}(1) + C$$

$$\frac{25}{2} - \frac{2}{3} = C$$

$$\frac{75}{6} - \frac{4}{6} = C$$

$$\frac{71}{6} = C$$

$$\frac{y^2}{2} = \frac{2}{3}e^{3x} + \frac{71}{6}$$

$$2\left(\frac{y^2}{2} = \frac{2}{3}e^{3x} + \frac{71}{6}\right)$$

$$y^2 = \frac{4}{3}e^{3x} + \frac{142}{6}$$

$$y = \pm\sqrt{\frac{4}{3}e^{3x} + \frac{71}{3}}$$

$$\boxed{y = \sqrt{\frac{4}{3}e^{3x} + \frac{71}{3}} \quad \text{since } y(0) = 5}$$

19

**Differential Equations Practice WS (1-3)**

1)

Given the differential equation $\frac{dy}{dx} = -\frac{2x}{y^2}$, find the particular solution, $y = f(x)$, with the initial condition $f(-1) = 3$.

A) $y = \sqrt{-2x+3}$

B) $y = \sqrt[3]{-3x^2+30}$

C) $y = \sqrt[3]{-3x^2+24}$

D) $y = \sqrt{-2x+7}$

E) $y = \sqrt{-3x^2-10}$

2) Given the differential equation $\frac{y'}{3-x} = 6y$ , find the particular solution, $y = f(x)$, with the initial condition f(0) = 2

A) $y = \sqrt{-\frac{3}{2}x^2 + x + 2}$

B) $y = \sqrt{-3x^2 + 36x + 4}$

C) $y = \ln|18x - 3x^2| + 2$

D) $y = e^{18x-3x^2} + 2$

E) $y = 2e^{18x-3x^2}$

20

3)

Given the differential equation $\frac{dy}{dx} = \frac{2x-1}{y}$, find the particular solution, $y = f(x)$, with the initial condition $f(-3) = 6$.

4)

What is the particular solution to the differential equation $\frac{dy}{dx} = x^2 y$ with the initial condition $y(3) = e$?

21

5)

Given the differential equation, $ww' = t^2sec^2(2t^3)$ , find the particular solution , $w = f(t)$, with the initial condition $w(0) = -4$.

6)

Given the differential equation, $y'xlnx - y = 0$ , find the particular solution , $y = f(x)$, with the initial condition $f(e) = e$

**Calculus** **Ch. 5.7 Notes** **Integrals of Inverse Trig Functions**

Recall Rules for Inverse Trig Derivatives:

| | | |
|---|---|---|
| $\frac{d}{dx}\arcsin u = \frac{u'}{\sqrt{1-u^2}}$<br>$\frac{d}{dx}\arctan u = \frac{u'}{1+u^2}$<br>$\frac{d}{dx}\operatorname{arc}\sec u = \frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$ | $\frac{d}{dx}\arccos u = -\frac{u'}{\sqrt{1-u^2}}$<br>$\frac{d}{dx}\operatorname{arc}\cot u = -\frac{u'}{1+u^2}$<br>$\frac{d}{dx}\operatorname{arc}\csc u = -\frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$ | **Inverse Trig Integral Rules**:<br>***a** is a constant*<br>1. $\int\frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right)+C$<br>2. $\int\frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$<br>3. $\int\frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arc}\sec\left(\frac{\lvert u\rvert}{a}\right)+C$ |

**Ex. 1:** $\int\frac{5}{x\sqrt{x^2-9}}dx$

**Ex. 2:** $\int\frac{1}{4+(x-1)^2}dx$

**Ex. 3:** $\int\frac{1}{\sqrt{7-16x^2}}dx$

24

Inverse Trig Integral Rules

*a is a constant* 1. $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right)+C$ 2. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$ 3. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right)+C$

Completing the Square Steps:

1. Write in standard form: $x^2 + bx + c$
2. Add spaces $x^2 + bx +$ ___ $+ c -$ ___
3. Put $\left(\frac{b}{2}\right)^2$ into the spaces
4. Factor expression

**Ex. 4:** $\int \frac{dx}{x^2-6x+13}$

**Ex. 5:** $\int \frac{2x-5}{x^2+2x+2}dx$

# AP Calculus AB Integration Technique Checklist

1) **Power Rule** (Can you rearrange problem to rely on just power rule?)

*Some examples include: $\int (3-x)^2 \left(\frac{2}{\sqrt{x}}\right) dx$ and $\int \frac{2x(5-3x+x^4)}{3(\sqrt{x^7})} dx$

a) convert radicals to rational exponential form (example: $\sqrt{x^5} = x^{\frac{5}{2}}$ )

b) move denominator variable to numerator

c) resolve parentheses and separate the terms.

*typically, if there are multiple terms in denominator separated by addition or subtraction, power rule alone will not be enough to make progress. Proceed to Option #2

2) If unable to rely on just power rule, then explore **U-Substitution** options.

a) Big picture: We want to choose a u-value that will lead to an exact match with a **known Integral rule.** (Needs to be a perfect match outside of coefficient terms, and with no x-variables remaining)

b) If expression can be rewritten using parentheses, the u-value is usually the expression inside the set of parentheses.

c) u-value is more than just replacing an "x", and may involve replacing a significant portion of the expression.

d) For fractional expressions, the u-value usually comes from the denominator.

(potential notable exceptions are log functions like $\ln x$ and radical expressions like $\sqrt{x}$ )

e) u-value are typically higher degree expressions when choosing between 2 expressions with different degrees.

2b) U-Substitution (using **change of variable**)

a) If the initial round of u-substitution is not enough to remove the remaining x's in the integrand, then explore option of rearranging the expression assigned to u, and solving for x.

b) Once we make that second set of substitutions, the problem is now purely in terms of u, and with all x's removed and replaced.

3) Rewrite rational expression using **Long Division** (synthetic division)

a) Condition needed to apply **long division** is the **numerator degree** $\geq$ **denominator degree.** (example: $\int \frac{2x^3-4x+1}{x^2+3} dx$ )

b) For long division problems, we can apply **synthetic division** only if denominator degree is = 1 (linear degree) (example: $\int \frac{4x^3-7x+2}{x-5} dx$ )

c) Once our rewrite is complete, we can typically find the antiderivative by using a combination of power rule and u-substitution across the different terms.

26

## Integration Technique Checklist (continued)

4) **ArcTrig Integral Rule** (From Ch. 5.7)

   a) If the **denominator degree > numerator degree by 2 or more degrees**, consider the ArcTrig Integral rules as potential match for the problem.

   b) If the "a" and "u" values of the ArcTrig Integral rule are not clearly visible, then apply the **completing the square method** in the denominator. This process will create a different (but equivalent) form that allows the "a" and "u" values in the denominator to become more visible.

   *Completing the square steps:

   i) Write expression in the form of $x^2 + bx + c$
   ii) Add spaces: $x^2 + bx + \_\_ + c - \_\_\_$
   iii) Insert $\left(\frac{b}{2}\right)^2$ into both the above ____ spaces
   iv) Factor, then identify the a-value(constant) and u-value(variable expression)
   v) Apply the ArcTrig Integral Rule

**AP Calculus AB** **Visual Comparison between Integral Rules (mostly Rational expressions)**

| Compare Numerator and Denominator to help determine Integral Rule(s) to apply | Example #1 | Example #2 |
| --- | --- | --- |
| 1) **Only 1 Term in the Denominator** (regardless of degree differences between numerator and denominator)<br><br>Solution: Consider expanding and splitting up the terms into individual fractions and applying integral rule for each term separately. | $\int \frac{x^4 - 5x^3 + 1}{2x^4} dx$ | $\int \frac{4e^{4x} - e^{2x}}{6e^{3x}} dx$ |
| 2) Multiple terms in the denominator and the **Denominator has variable exponent degree that is 1 higher than the Numerator**<br><br>Solution: Consider U-Substitution | $\int \frac{5x}{7x^2 - 4} dx$ | $\int \frac{2x^2}{\sqrt[5]{3x^3 - 4}} dx$ |
| 3) Multiple terms in the denominator and the **Numerator has variable exponent that is Same degree OR Higher than the Denominator.**<br><br>Solution: Consider Long Division and/or Synthetic Division | $\int \frac{4x - 3}{x - 5} dx$<br><br>Apply long division or synthetic division | $\int \frac{x^4 + x - 4}{x^2 + 2} dx$<br><br>Apply long division (synthetic division does **not** apply) |
| 4) Multiple terms in the denominator and the **Denominator has variable exponent that is higher than the Numerator by 2 or more degrees**:<br><br>Solution: Consider ArcTrig Integral Rules | $\int \frac{1}{x^2 - 8x + 4} dx$<br><br>Apply Arctan Integral Rule | $\int \frac{5x}{\sqrt{1 - x^4}} dx$<br><br>Apply Arcsin Integral Rule |

**AP Calculus AB** **Visual Comparison between Integral Rules (mostly Rational expressions)** Key

| Compare Numerator and Denominator to help determine Integral Rule(s) to apply | Example #1 | Example #2 |
|---|---|---|

**1) Only 1 Term in the Denominator** (regardless of degree differences between numerator and denominator)

Solution: Consider expanding and splitting up the terms into individual fractions and applying integral rule for each term separately.

Example #1:

$$\int \frac{x^4-5x^3+1}{2x^4}\,dx$$

$\int \frac{x^4}{2x^4} - \frac{5x^3}{2x^4} + \int \frac{1}{2x^4}\,dx$

$\int \frac{1}{2} - \frac{5}{2}\left(\frac{1}{x}\right) + \frac{1}{2}x^{-4}\,dx$

$\boxed{\frac{1}{2}x - \frac{5}{2}\ln|x| + \frac{1}{2}\left(\frac{x^{-3}}{-3}\right) + C}$

Example #2:

$$\int \frac{4e^{4x}-e^{2x}}{6e^{3x}}\,dx$$

$\int \frac{4e^{4x}}{6e^{3x}} - \frac{e^{2x}}{6e^{3x}}\,dx \rightarrow \int \frac{2}{3}e^{x} - \frac{1}{6}e^{-x}\,dx$

$\boxed{\frac{2}{3}e^{x} + \frac{1}{6}e^{-x} + C}$

**2) Multiple terms in the denominator and the Denominator has variable exponent degree that is 1 higher than the Numerator**

Solution: Consider U-Substitution

Example #1:

$$\int \frac{5x}{7x^2-4}\,dx$$

$u = 7x^2-4$

$\frac{du}{dx} = 14x$

$14x\,dx = du$

$dx = \frac{du}{14x}$

$\int \frac{5x}{u}\cdot\frac{du}{14x}$

$\frac{5}{14}\int \frac{1}{u}\,du$

$\boxed{\frac{5}{14}\ln|7x^2-4| + C}$

Example #2:

$$\int \frac{2x^2}{\sqrt[5]{3x^3-4}}\,dx \rightarrow \int \frac{2x^2}{(3x^3-4)^{1/5}}\,dx$$

$u = 3x^3-4$

$\frac{du}{dx} = 9x^2$

$dx = \frac{du}{9x^2}$

$\int \frac{2x^2}{u^{1/5}}\cdot\frac{du}{9x^2}$

$\frac{2}{9}\int u^{-1/5}\,du$

$\frac{2}{9}\left(\frac{u^{4/5}}{4/5}\right) + C$

$\frac{10}{36}u^{4/5} + C$

$\boxed{\frac{5}{18}(3x^3-4)^{4/5} + C}$

**3) Multiple terms in the denominator and the Numerator has variable exponent that is Same degree OR Higher than the Denominator.**

Solution: Consider Long Division and/or Synthetic Division

Example #1:

$$\int \frac{4x-3}{x-5}\,dx$$

Apply long division or synthetic division

$$\begin{array}{r|rr} 5 & 4 & -3 \\ & \downarrow & 20 \\ \hline & 4 & 17 \end{array}$$

$\int 4 + \frac{17}{x-5}\,dx$

$\boxed{4x + 17\ln|x-5| + C}$

Example #2:

$$\int \frac{x^4+x-4}{x^2+2}\,dx$$

Apply long division (synthetic division does not apply)

u-sub: $u = x^2+2$, $\frac{du}{dx} = 2x$, $dx = \frac{du}{2x}$

$$\begin{array}{r} x^2 - 2 + \frac{x}{x^2+2} \\ x^2+2\,\overline{)\,x^4 + x - 4} \\ -(x^4 + 2x^2) \\ \hline -2x^2 + x - 4 \\ -(-2x^2 - 4) \\ \hline x \end{array}$$

$x \leftarrow$ Remainder

$\int x^2 - 2 + \frac{x}{x^2+2}\,dx$

$\int \frac{x}{u}\cdot\frac{du}{2x}$

$\frac{1}{2}\int \frac{1}{u}\,du$

$\boxed{\frac{x^3}{3} - 2x + \frac{1}{2}\ln|x^2+2| + C}$

**4) Multiple terms in the denominator and the Denominator has variable exponent that is higher than the Numerator by 2 or more degrees:**

Solution: Consider ArcTrig Integral Rules

$x^2 - 8x + 4$

$\left(\frac{b}{2}\right)^2 \rightarrow \left(\frac{8}{2}\right)^2 = 16$

$x^2 - 8x + \underline{16} + 4 + \underline{16}$

$(x-4)^2 + 20$

Example #1:

$$\int \frac{1}{x^2-8x+4}\,dx$$

Apply Arctan Integral Rule

$\int \frac{dx}{(x-4)^2 + (\sqrt{20})^2}$

$u = x-4$

$a = \sqrt{20}$

$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$

$\frac{1}{\sqrt{20}}\arctan\left(\frac{x-4}{\sqrt{20}}\right) + C$

Example #2:

$$\int \frac{5x}{\sqrt{1-x^4}}\,dx$$

$\frac{5}{2}\arcsin\left(\frac{x^2}{1}\right) + C$

Apply Arcsin Integral Rule

$\int \frac{5x}{\sqrt{(\ )^2 - (\ )^2}}\,dx$

$\int \frac{5x}{\sqrt{(1)^2 - (x^2)^2}}\,dx$

$a = 1$

$u = x^2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$\int \frac{5x}{\sqrt{a^2-u^2}}\cdot\frac{du}{2x}$

$\frac{5}{2}\int \frac{du}{\sqrt{a^2-u^2}}$

$\boxed{\frac{5}{2}\arcsin(x^2) + C}$