

6.3 Notes: Differential Equations

1/3

Separation of Variables: Rearrange equation with y and dy on the left and x 's, dx on the right of equation.

Ex. 1 $\frac{dy}{dx} = \frac{2x}{y}$

*dy and dx are always in the numerator

$$\rightarrow y dy = 2x dx$$

$$\int y dy = \int 2x dx$$

$$\frac{y^2}{2} = \frac{2x^2}{2} + C$$

$$2\left(\frac{y^2}{2}\right) = 2\left(\frac{2x^2}{2} + C\right)$$

$$\boxed{y^2 = 2x^2 + C}$$

you only need to write "+C" once on the right side of equation.

Ex. 2 $\frac{dy}{dx} = x(1+y)$

$$\int \frac{dy}{1+y} = \int x dx$$

$$\log|1+y| = \frac{x^2}{2} + C$$

$$e^{\frac{x^2}{2} + C} = 1+y$$

This is a constant

$$u = 1+y \quad \int \frac{1}{u} du$$

$$\frac{du}{dy} = 1 \quad dy = du$$

$$\ln|1+y| = \frac{x^2}{2} + C$$

$$\ln|1+y| = \frac{x^2}{2} + C$$

$$e^{\ln|1+y|} = e^{\frac{x^2}{2} + C}$$

$$|1+y| = e^{\frac{x^2}{2} + C}$$

recall exponential property
 $a^m \cdot a^n = a^{m+n}$

$$|1+y| = e^{\frac{x^2}{2}} \cdot e^C$$

$$|1+y| = e^{\frac{x^2}{2}} \cdot C$$

$$\boxed{y = Ce^{\frac{x^2}{2}} - 1}$$

* Absolute value sign drops away b/c "C" represents both positive and negative values.

6.3 Notes (continued)

2/3

Ex.3 Find a general solution of $2x + 3yy' = 0$. Then find the particular solution, $y = f(x)$, if the solution passes through the point $(1, -2)$

*Steps

- 1) Rewrite y' as $\frac{dy}{dx}$
- 2) Separate Variables
- 3) Solve (Indefinite Integral)
- 4) Find specific solution (find "C")

$$2x + 3y y' = 0$$

$$2x + 3y \left(\frac{dy}{dx}\right) = 0$$

$$3y \left(\frac{dy}{dx}\right) = -2x$$

$$y dy = -\frac{2}{3}x dx$$

$$\int y dy = \int -\frac{2}{3}x dx$$

$$\frac{y^2}{2} = -\frac{2}{3} \left(\frac{x^2}{2}\right) + C \rightarrow \frac{y^2}{2} = -\frac{1}{3}x^2 + C$$

$$y^2 = -\frac{2}{3}x^2 + C \quad \begin{matrix} \text{solve for "C"} \\ \text{using } (1, -2) \end{matrix}$$

$$(-2)^2 = -\frac{2}{3}(1)^2 + C$$

$$4 = -\frac{2}{3} + C$$

$$4 + \frac{2}{3} = C$$

$$\frac{14}{3} = C$$

dy and dx always
in numerator

$$y^2 = -\frac{2}{3}x^2 + \frac{14}{3}$$

$$y = \pm \sqrt{-\frac{2}{3}x^2 + \frac{14}{3}}$$

$$-2 = +\sqrt{-\frac{2}{3}(1)^2 + \frac{14}{3}}$$

$$-2 \neq +\sqrt{4}$$

$$-2 = -\sqrt{-\frac{2}{3}(1)^2 + \frac{14}{3}}$$

$$-2 = -\sqrt{4}$$

$$-2 = -2 \checkmark$$

*These are 2 separate equations.
Test which equation
will pass through
given point $(1, -2)$

The particular equation
that will pass through $(1, -2)$
is therefore

$$y = -\sqrt{-\frac{2}{3}x^2 + \frac{14}{3}}$$

6.3 Notes (continued)

3/3

Ex. 4 Find a general solution to $yy' = 6\cos(\pi x)$. Then find the particular solution, $y=f(x)$ if the function passes through point $(1, 2)$

$$yy' = 6\cos(\pi x)$$

$$y \frac{dy}{dx} = 6\cos(\pi x)$$

$$y dy = 6\cos(\pi x) dx$$

$$\int y dy = \int 6\cos(\pi x) dx$$

$$u = \pi x \quad du = \frac{d}{\pi} dx$$

$$\frac{du}{dx} = \pi \quad \int 6\cos u \cdot \frac{du}{\pi}$$

$$\int \frac{6}{\pi} \cos u du$$

$$\frac{y^2}{2} = \frac{6}{\pi} \sin u + C \quad \text{solve for "C" using } (1, 2)$$

$$y^2 = \frac{12}{\pi} \sin(\pi x) + C$$

$$2^2 = \frac{12}{\pi} \sin(\pi) + C$$

$$4 = \frac{12}{\pi}(0) + C$$

$$4 = C$$

$$y^2 = \frac{12}{\pi} \sin(\pi x) + 4$$

$$y^2 = \frac{12}{\pi} \sin(\pi x) + 4$$

$$y = \pm \sqrt{\frac{12}{\pi} \sin(\pi x) + 4}$$

*Test both equations to determine function passing through point $(1, 2)$

$$y = +\sqrt{\frac{12}{\pi} \sin(\pi x) + 4}$$

$$2 = +\sqrt{\frac{12}{\pi} \sin(\pi) + 4}$$

$$2 = \sqrt{\frac{12}{\pi}(0) + 4}$$

$$2 = \sqrt{4}$$

$$2 = 2 \checkmark$$

$$y = -\sqrt{\frac{12}{\pi} \sin(\pi x) + 4}$$

$$2 = -\sqrt{\frac{12}{\pi} \sin(\pi) + 4}$$

$$2 = -\sqrt{\frac{12}{\pi}(0) + 4}$$

$$2 = -\sqrt{4}$$

$$2 \neq -2$$

$$y = +\sqrt{\frac{12}{\pi} \sin(\pi x) + 4}$$

Ex. 5

Solve differential equation: $y' = (x+1)y$

$$y' = (x+1)y$$

$$\frac{dy}{dx} = (x+1)y$$

$$\frac{dy}{y} = (x+1)dx$$

$$\int \frac{dy}{y} = \int (x+1)dx$$

$$\ln|y| = \frac{x^2}{2} + x + C$$

$$e^{\ln|y|} = e^{\frac{x^2}{2} + x + C}$$

$$|y| = e^{\frac{x^2}{2} + x} \cdot e^C$$

$$|y| = e^{\frac{x^2}{2} + x} \cdot C$$

$$|y| = Ce^{\frac{x^2}{2} + x}$$

$$y = Ce^{\frac{x^2}{2} + x}$$

* Absolute value sign drops away because "C" represents both positive and negative values

6.3 Notes continued

- 4) Find a general solution to $yy' = 6\cos(\pi x)$. Then find the particular solution, $y = f(x)$, if the function passes through the point $(1, 2)$.

$$yy' = 6\cos(\pi x)$$

$$y \left(\frac{dy}{dx} \right) = 6\cos(\pi x)$$

$$\frac{y dy}{dx} = 6\cos(\pi x)$$

$$\int y dy = \int 6\cos(\pi x) dx$$

$$\frac{y^2}{2} = \int 6\cos u \cdot \frac{du}{\pi}$$

$$\frac{y^2}{2} = 6 \cdot \frac{1}{\pi} \int \cos u du$$

$$\frac{y^2}{2} = \frac{6}{\pi} \sin u + C$$

$$2 \left(\frac{y^2}{2} \right) = \frac{6}{\pi} \sin(\pi x) + C$$

$$y^2 = \frac{12}{\pi} \sin(\pi x) + C$$

use $(1, 2)$
solve for C

$$2^2 = \frac{12}{\pi} \sin(\pi(1)) + C$$

$$4 = \frac{12}{\pi}(0) + C$$

$$4 = C$$

point $(1, 2)$

$$\sqrt{y^2 + \frac{12}{\pi} \sin(\pi x) + 4}$$

$$y = \sqrt{\frac{12}{\pi} \sin(\pi x) + 4}$$

~~$$y = -\sqrt{\frac{12}{\pi} \sin(\pi x) + 4}$$~~

- 5) Solve $y' = (x+1)y$

5) General equation for $y' = (x+1)y$ * $e^{\ln x} = x$

$$\frac{dy}{dx} = \frac{(x+1)y}{1}$$

$$dy = (x+1)y dx$$

$$\frac{dy}{y} = (x+1)dx$$

$$\ln|y| = \frac{x^2}{2} + x + C$$

$$|y| = e^{\frac{x^2}{2} + x} [e^C] \leftarrow C$$

$$|y| = Ce^{\frac{x^2}{2} + x}$$

$$y = \pm Ce^{\frac{x^2}{2} + x}$$

~~$$y dy = (x+1)dx$$~~

$$\int \frac{1}{y} dy = \int (x+1) dx$$

$$\ln|y| = \frac{x^2}{2} + x + C$$

6.3 Homework p.429 #1-21 odd
 Differential Equations

Find the general solution of differential equation.

$$9) \sqrt{1-4x^2} y' = x$$

$$\frac{dy}{dx} \sqrt{1-4x^2} = x$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{1-4x^2}}$$

$$\int dy = \int \frac{x}{\sqrt{1-4x^2}} dx$$

u-substitution

$$\begin{cases} u = 1-4x^2 \\ \frac{du}{dx} = -8x \\ dx = \frac{du}{-8x} \end{cases}$$

$$\int \frac{x}{\sqrt{u}} \cdot \frac{du}{-8x} = -\frac{1}{8} \int \frac{1}{\sqrt{u}} du$$

$$= -\frac{1}{8} \int u^{-1/2} du = -\frac{1}{8} \left(\frac{u^{1/2}}{1/2} \right)$$

$$= -\frac{1}{4} u^{1/2}$$

$$y = \frac{-1}{4} u^{1/2} + C$$

$$\boxed{y = \frac{-1}{4} \sqrt{1-4x^2} + C}$$

$$11) y \ln x - xy' = 0$$

$$y \ln x - x \left(\frac{dy}{dx} \right) = 0$$

$$-x \left(\frac{dy}{dx} \right) = -y \ln x$$

$$\frac{dy}{y} = \frac{\ln x}{x} dx$$

$$\int \frac{dy}{y} = \int \frac{\ln x}{x} dx$$

$$\begin{aligned} u &= \ln x \\ \frac{du}{dx} &= \frac{1}{x} \quad dx = x du \end{aligned}$$

$$\ln|y| = \int \frac{u}{x} \cdot x du$$

$$= \frac{u^2}{2} + C$$

$$\ln|y| = \frac{(\ln x)^2}{2} + C$$

$$e^{\ln|y|} = e^{\frac{(\ln x)^2}{2} + C}$$

remember $a^m \cdot a^n = a^{m+n}$

$$y = e^{\frac{(\ln x)^2}{2} + C}$$

this is a constant

$$y = e^{\frac{(\ln x)^2}{2}} \cdot C$$

$$\boxed{y = Ce^{\frac{(\ln x)^2}{2}}}$$

6.3 Homework (continued)

Find the particular solution satisfying the initial condition.

13) $y y' - e^x = 0$ given $y(0) = 4$

$$y \left(\frac{dy}{dx} \right) - e^x = 0$$

$$y \left(\frac{dy}{dx} \right) = e^x$$

$$\int y dy = \int e^x dx$$

$$\frac{y^2}{2} = e^x + C$$

$$y^2 = 2e^x + C$$

$$4^2 = 2e^0 + C$$

$$16 = 2 + C$$

$$\underline{14 = C}$$

$$y^2 = 2e^x + 14$$

$$y = \pm \sqrt{2e^x + 14}$$

test which equation will satisfy $(0, 4)$

$$y = +\sqrt{2e^x + 14}$$

15) $y(x+1) + y' = 0$ given $y(-2) = 1$

$$y(x+1) + \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -y(x+1)$$

$$\frac{dy}{y} = -(x+1)dx$$

$$\int \frac{dy}{y} = - \int (x+1) dx$$

$$\ln|y| = -\frac{x^2}{2} - x + C$$

$$y = e^{-\frac{x^2}{2} - x} \boxed{e^C} = \text{constant}$$

$$y = e^{-\frac{x^2}{2} - x} \cdot C$$

$$y = C e^{-\frac{x^2}{2} - x} \quad \begin{matrix} \text{plug in } (-2, 1) \\ \text{to find "C"} \end{matrix}$$

$$1 = C e^{-\frac{4}{2} + 2}$$

$$1 = C e^0$$

$$1 = C$$

$$y = 1 \cdot e^{-\frac{x^2}{2} - x}$$

$$y = e^{-\frac{x^2}{2} - x}$$

6.3 (Homework continued)

(7) $y(1+x^2)y' - x(1+y^2) = 0$ given $y(0) = \sqrt{3}$

$$y(1+x^2)\frac{dy}{dx} - x(1+y^2) = 0$$

$$\int \frac{y}{1+y^2} dy = \int \frac{x}{1+x^2} dx$$

$$\begin{aligned} u &= 1+y^2 & u &= 1+x^2 \\ \frac{du}{dy} &= 2y & \frac{du}{dx} &= 2x \\ dy &= \frac{du}{2y} & dx &= \frac{du}{2x} \end{aligned}$$

$$\int \frac{1}{u} \cdot \frac{du}{2y} = \int \frac{x}{u} \cdot \frac{du}{2x}$$

$$\frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \int \frac{1}{u} du$$

$$\frac{1}{2} \ln|1+y^2| = \frac{1}{2} \ln|1+x^2| + C$$

$$\ln|1+y^2| = \ln|1+x^2| + C$$

$$1+y^2 = e^{\ln|1+x^2|+C}$$

$$1+y^2 = (1+x^2) \cdot C \rightarrow 1+y^2 = C(1+x^2)$$

(9) $\frac{du}{dv} = uv \sin v^2$ at $u(0) = 1$

$$\int \frac{du}{u} = \int v \sin(v^2) dv$$

$$\begin{aligned} u &= v^2 & dv &= \frac{du}{2v} \\ \frac{du}{u} &= \frac{1}{2} \int v \sin(u) \frac{du}{v} \end{aligned}$$

$$\ln|u| = \frac{1}{2}(-\cos(v^2)) + C$$

$$\ln|u| = \frac{1}{2}\cos v^2 + C$$

$$u = e^{\frac{1}{2}\cos v^2 + C}$$

\boxed{C} this is a constant

*plug in $(0, 1)$ to find "C"

$$1 = Ce^{-\frac{1}{2}\cos(0)^2}$$

$$1 = Ce^{-\frac{1}{2}(1)}$$

*plug in $(0, \sqrt{3})$ to find "C"

$$1 + (\sqrt{3})^2 = C(1+0^2) \quad 4 = C$$

$$1+y^2 = 4(1+x^2)$$

$$y^2 = 4+4x^2 - 1 \quad y^2 = 3+4x^2$$

$$y = \pm \sqrt{3+4x^2} \rightarrow \boxed{y = \sqrt{3+4x^2}}$$

*choose equation that satisfies $(0, \sqrt{3})$

$$1 = \frac{C}{e^{1/2}}$$

$$e^{1/2} = C$$

$$u = e^{\frac{1}{2}-\frac{1}{2}\cos v^2}$$

$$\boxed{u = e^{\frac{1}{2}-\frac{1}{2}\cos v^2}}$$

6.3 Homework (continued)

21) $dP - kPdt = 0$ given: $P(0) = P_0$

$$dP = kPdt$$

$$\int \frac{dP}{P} = \int kdt$$

$$\ln|P| = kt + C$$

$$\begin{aligned} P &= e^{kt+C} \\ P &= Ce^{kt} \\ * \text{ plug in } (0, P_0) \text{ to find "C"} \\ P_0 &= Ce^{k(0)} \\ P_0 &= Ce^0 \\ \underline{\underline{P_0 = C}} \end{aligned}$$

$$P = P_0 e^{kt}$$

6.3 Day 1
Classwork
Key

$$1) \frac{dy}{dx} = \frac{2x}{y}$$

$$y dy = 2x dx$$

$$\int y dy = \int 2x dx$$

$$2\left(\frac{y^2}{2} = \frac{2x^2}{2} + C\right)$$

$$y^2 = 2x^2 + 2C$$

$$y^2 = dx^2 + C$$

general solution:

$$2) \frac{dy}{dx} = \frac{x(1+y)}{1}$$

$$\frac{dy}{1+y} = \frac{x(1+y)}{1} dx$$

$$\int \frac{dy}{1+y} = \int x dx$$

$$\ln|1+y| = \frac{x^2}{2} + C \quad a^{m+n} = a^m \cdot a^n$$

$$\int \frac{1}{1+y} dy = \int x dx$$

$$|1+y| = e^{\frac{x^2}{2}} \boxed{e^c} < C$$

$$\ln|1+y| = \frac{x^2}{2} + C$$

$$u = 1+y \quad | dy = du$$

$$\frac{du}{dy} = 1 \quad \left| \int u du = \ln|u| \right.$$

$$\ln|1+y|$$

$$|1+y| = Ce^{\frac{x^2}{2}}$$

$$1+y = \pm Ce^{\frac{x^2}{2}}$$

$$1+y = Ce^{\frac{x^2}{2}}$$

$$\boxed{y = Ce^{\frac{x^2}{2}} - 1}$$

General solution.

$$3) \quad 2x + 3y y' = 0$$

*particular solution (1, -2)

$$2x + 3y \left(\frac{dy}{dx} \right) = 0$$

$$3y \left(\frac{dy}{dx} \right) = -2x$$

$$\frac{3y dy}{dx} = \frac{-2x}{1}$$

$$\frac{3y dy}{3} = \frac{-2x dx}{3}$$

$$\int y dy = \int \frac{-2}{3} x dx$$

$$\frac{y^2}{2} = \frac{-2}{3} \cdot \frac{x^2}{2} + C$$

$$\left(\frac{y^2}{2} \right) = \frac{-1}{3} x^2 + C$$

$$y^2 = \frac{-2}{3} x^2 + C$$

$$y^2 = \frac{-2}{3} x^2 + C$$

$$(-2)^2 = \frac{-2}{3}(1)^2 + C$$

$$4 = \frac{-2}{3} + C$$

$$\frac{12}{3} + \frac{2}{3} = C$$

$$\frac{14}{3} = C$$

$$\sqrt{y^2} = \sqrt{\frac{-2}{3} x^2 + \frac{14}{3}}$$

$$y = \sqrt{\frac{-2}{3} x^2 + \frac{14}{3}}$$

$$\boxed{y = \sqrt{\frac{-2}{3} x^2 + \frac{14}{3}}}$$

solve for C
using (1, -2)

6.3 Homework p. 421 #1-21 odd

Find general solutions of differential equation

$$1) \frac{dy}{dx} = \frac{x}{y}$$

$$y dy = x dx$$

$$\int y dy = \int x dx$$

$$\begin{aligned} \frac{y^2}{2} &= \frac{x^2}{2} + C \\ 2\left(\frac{y^2}{2} = \frac{x^2}{2} + C\right) &\\ \boxed{y^2 = x^2 + C} & \end{aligned}$$

$$3) x^2 + 5y \left(\frac{dy}{dx} \right) = 0$$

$$5y \left(\frac{dy}{dx} \right) = -x^2$$

$$\frac{5y dy}{dx} = -x^2$$

$$5y dy = -x^2 dx$$

$$y dy = -\frac{x^2}{5} dx$$

$$y dy = -\frac{1}{5} x^2 dx$$

$$\int y dy = -\frac{1}{5} \int x^2 dx$$

$$\frac{y^2}{2} = -\frac{1}{5} \cdot \frac{x^3}{3} + C$$

$$\frac{y^2}{2} = -\frac{1}{15} x^3 + C$$

$$2 \left[\frac{y^2}{2} = -\frac{1}{15} x^3 + C \right]$$

$$\boxed{y^2 = -\frac{2}{15} x^3 + C}$$

$$5) \frac{dr}{ds} = 0.75r$$

$$dr = 0.75r ds$$

$$\frac{dr}{r} = 0.75 ds$$

$$\int \frac{dr}{r} = \int 0.75 ds$$

$$\int \frac{1}{r} dr = \int 0.75 ds$$

$$\ln|r| = 0.75s + C$$

$$e^{\ln|r|} = e^{0.75s + C}$$

$$|r| = e^{0.75s} \cdot e^C$$

$$|r| = e^{0.75s} \cdot C$$

$$|r| = Ce^{0.75s}$$

$$r = \pm Ce^{0.75s}$$

$$\boxed{r = Ce^{0.75s}}$$

$$7) (2+x)y' = 3y$$

$$(2+x)\frac{dy}{dx} = 3y$$

$$(2+x)dy = 3y dx$$

$$\frac{(2+x)dy}{y} = 3dx$$

$$\frac{dy}{y} = \frac{3}{2+x} dx$$

$$\int \frac{1}{y} dy = \int \frac{3}{2+x} dx$$

$$\begin{aligned} u &= 2+x & dx &= du \\ \frac{du}{dx} &= 1 \\ \int \frac{3}{u} du & \end{aligned}$$

$$|\ln|y|| = 3\ln|2+x| + C$$

$$|\ln|y|| = \ln|2x+1|^3 + C$$

$$e^{\ln|y|} = e^{\ln|2x+1|^3 + C}$$

$$|y| = e^{\ln|2x+1|^3} \cdot e^C$$

$$|y| = |2x+1|^3 \cdot C$$

$$|y| = C|2x+1|^3$$

$$y = \pm C(2x+1)^3$$

$$\boxed{y = C(2x+1)^3}$$

$$9) yy' = 4\sin x$$

$$y\left(\frac{dy}{dx}\right) = 4\sin x$$

$$ydy = 4\sin x dx$$

$$\int ydy = \int 4\sin x dx$$

$$\frac{y^2}{2} = 4(-\cos x) + C$$

$$2\left[\frac{y^2}{2} = -4\cos x + C\right]$$

$$\boxed{y^2 = -8\cos x + C}$$

$$y = -\frac{1}{4}u^{1/2} + C$$

$$\boxed{y = -\frac{1}{4}(1-4x^2)^{1/2} + C}$$

$$11) \sqrt{1-4x^2} y' = x$$

$$\sqrt{1-4x^2} \cdot \frac{dy}{dx} = x$$

$$\sqrt{1-4x^2} dy = x dx$$

$$dy = \frac{x}{\sqrt{1-4x^2}} dx$$

$$\int dy = \int \frac{x}{\sqrt{1-4x^2}} dx$$

$$\int x \cdot u^{-1/2} \cdot \frac{du}{-8x}$$

$$\int x(1-4x^2)^{-1/2} dx$$

$$-\frac{1}{8} \int u^{-1/2} du$$

$$u = 1-4x^2$$

$$\frac{du}{dx} = -8x$$

$$dx = \frac{du}{-8x}$$

$$-\frac{1}{8} \cdot \frac{u^{1/2}}{1/2}$$

$$-\frac{1}{8} \cdot 2u^{1/2}$$

$$13) y \ln x - xy' = 0$$

$$y \ln x - x \left(\frac{dy}{dx} \right) = 0$$

$$-x \frac{dy}{dx} = -y \ln x$$

$$xdy = y \ln x dx$$

$$\frac{dy}{y} = \frac{\ln x}{x} dx$$

$$\int \frac{1}{y} dy = \int \frac{\ln x}{x} dx$$

$$u = \ln x \quad dx = x du$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$\int \frac{u}{x} x du$$

$$|\ln y| = \frac{u^2}{2} + C$$

$$|\ln y| = \frac{1}{2} (\ln x)^2 + C$$

$$e^{|\ln y|} = e^{\frac{(\ln x)^2}{2} + C}$$

$$|y| = e^{\frac{(\ln x)^2}{2}} \cdot e^C$$

$$|y| = e^{\frac{(\ln x)^2}{2}} \cdot C$$

$$y = \pm C e^{\frac{(\ln x)^2}{2}}$$

$$\boxed{y = Ce^{\frac{(\ln x)^2}{2}}}$$

$$15) yy' - 2e^x = 0 \quad y(0) = 3 \quad \text{Write particular solution.}$$

$$y \left(\frac{dy}{dx} \right) = 2e^x$$

$$\frac{y^2}{2} = 2e^x + C$$

$$y dy = 2e^x dx$$

$$2 \left[\frac{y^2}{2} = 2e^x + C \right]$$

$$\int y dy = 2 \int e^x dx$$

$$y^2 = 4e^x + C$$

$$3^2 = 4e^0 + C$$

$$9 = 4 + C$$

$$5 = C$$

$$y^2 = 4e^x + 5$$

$$y = \pm \sqrt{4e^x + 5}$$

$$\boxed{y = \sqrt{4e^x + 5}}$$

$$y = -\sqrt{4e^x + 5}$$

* solve for
C: (0, 3)

$$17) y(x+1) + y' = 0 \quad \text{Initial Condition: } y(-2) = 1$$

$$\begin{array}{l|l}
 \begin{array}{l}
 y(x+1) + \frac{dy}{dx} = 0 \\
 \frac{dy}{dx} = -y(x+1) \\
 dy = -y(x+1)dx \\
 \frac{dy}{y} = -(x+1)dx
 \end{array} &
 \begin{array}{l}
 \int \frac{1}{y} dy = \int -x-1 dx \\
 \ln|y| = \frac{-x^2}{2} - x + C \\
 \ln|1| = \frac{-(\cdot)^2}{2} - (-2) + C \\
 0 = -2 + 2 + C \\
 0 = C
 \end{array} \\
 &
 \begin{array}{l}
 \text{* solve for } C \\
 \text{plug in } (-2, 1) \\
 e^{\ln|y|} = e^{\frac{-x^2}{2} - x} \\
 |y| = e^{\frac{-x^2}{2} - x} \\
 y = \pm e^{\frac{-x^2}{2} - x} \\
 \boxed{y = e^{\frac{-x^2}{2} - x}}
 \end{array}
 \end{array}$$

$$19) y(1+x^2)y' - x(1+y^2) = 0 \quad \text{Initial Condition: } y(0) = \sqrt{3}$$

$$\begin{array}{l|l}
 \begin{array}{l}
 y(1+x^2)\frac{dy}{dx} - x(1+y^2) = 0 \\
 y(1+x^2)\frac{dy}{dx} = x(1+y^2) \\
 y(1+x^2)dy = x(1+y^2)dx \\
 \frac{ydy}{1+y^2} = \frac{x dx}{1+x^2}
 \end{array} &
 \begin{array}{l}
 \frac{1}{2} \int \frac{1}{u} du \quad ; \quad \frac{1}{2} \int \frac{1}{u} du \\
 2 \left[\frac{1}{2} \ln|1+y^2| \right] = \frac{1}{2} \ln|1+x^2| + C \\
 \ln|1+y^2| = \ln|1+x^2| + C \\
 e^{\ln|1+y^2|} = e^{\ln|1+x^2| + C} \\
 |1+y^2| = e^{\ln|1+x^2|} \cdot e^C \\
 |1+y^2| = |1+x^2| \cdot C
 \end{array} \\
 &
 \begin{array}{l}
 1+y^2 = 4(1+x^2) \\
 1+y^2 = 4+4x^2 \\
 y^2 = 4x^2 + 3 \\
 y = \pm \sqrt{4x^2 + 3} \\
 \hline
 y = \sqrt{4x^2 + 3} \\
 \text{or} \\
 \boxed{y = \sqrt{4x^2 + 3}}
 \end{array} \\
 &
 \begin{array}{l}
 \text{* Equation} \\
 \text{satisfies ordered} \\
 \text{pair } (0, \sqrt{3})
 \end{array}
 \end{array}$$

$$21) \frac{du}{dv} = uv \sin(v^2) \quad \text{Initial Condition: } u(0) = 1$$

$$du = uv \sin(v^2) dv$$

$$\frac{du}{u} = v \sin(v^2) dv$$

$$\int \frac{1}{u} du = \int v \sin(v^2) dv$$

$$\begin{aligned} \ln|u| &= \int \frac{du}{u} \\ &\stackrel{u=v^2}{=} \int \frac{du}{2v} \\ &\stackrel{dv=\frac{du}{2v}}{=} \int v \sin(u) \cdot \frac{du}{2v} \\ &= \frac{1}{2} \int \sin(u) du \end{aligned}$$

$$\ln|u| = \frac{1}{2}(-\cos(v^2)) + C$$

$$e^{\ln|u|} = e^{-\frac{1}{2}\cos(v^2) + C}$$

$$|u| = e^{-\frac{1}{2}\cos(v^2)} \cdot e^C$$

$$|u| = e^{-\frac{1}{2}\cos(v^2)} \cdot C$$

$$|u| = Ce^{-\frac{1}{2}\cos(v^2)} \quad \leftarrow \begin{matrix} * \text{ plug in} \\ \text{point } (0, 1) \end{matrix}$$

$$| = Ce^{-\frac{1}{2}\cos 0}$$

$$| = Ce^{-\frac{1}{2}}$$

$$\frac{1}{e^{-\frac{1}{2}}} = C$$

$$\underline{e^{\frac{1}{2}} = C}$$

$$u = e^{\frac{1}{2}} \cdot e^{-\frac{1}{2}\cos(v^2)}$$

$$u = e^{\frac{1}{2} - \frac{1}{2}\cos(v^2)}$$

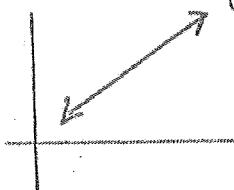
6.2 Homework (continued)

Determine if statement is True or False.

- 73) In exponential growth, rate is constant \rightarrow FALSE

$$y = Ce^{kt} \quad y' = C \cdot e^{kt} \cdot k \quad \text{is not a constant with variable existing in derivative function.}$$

- 74) In linear growth, rate of growth is constant \rightarrow TRUE



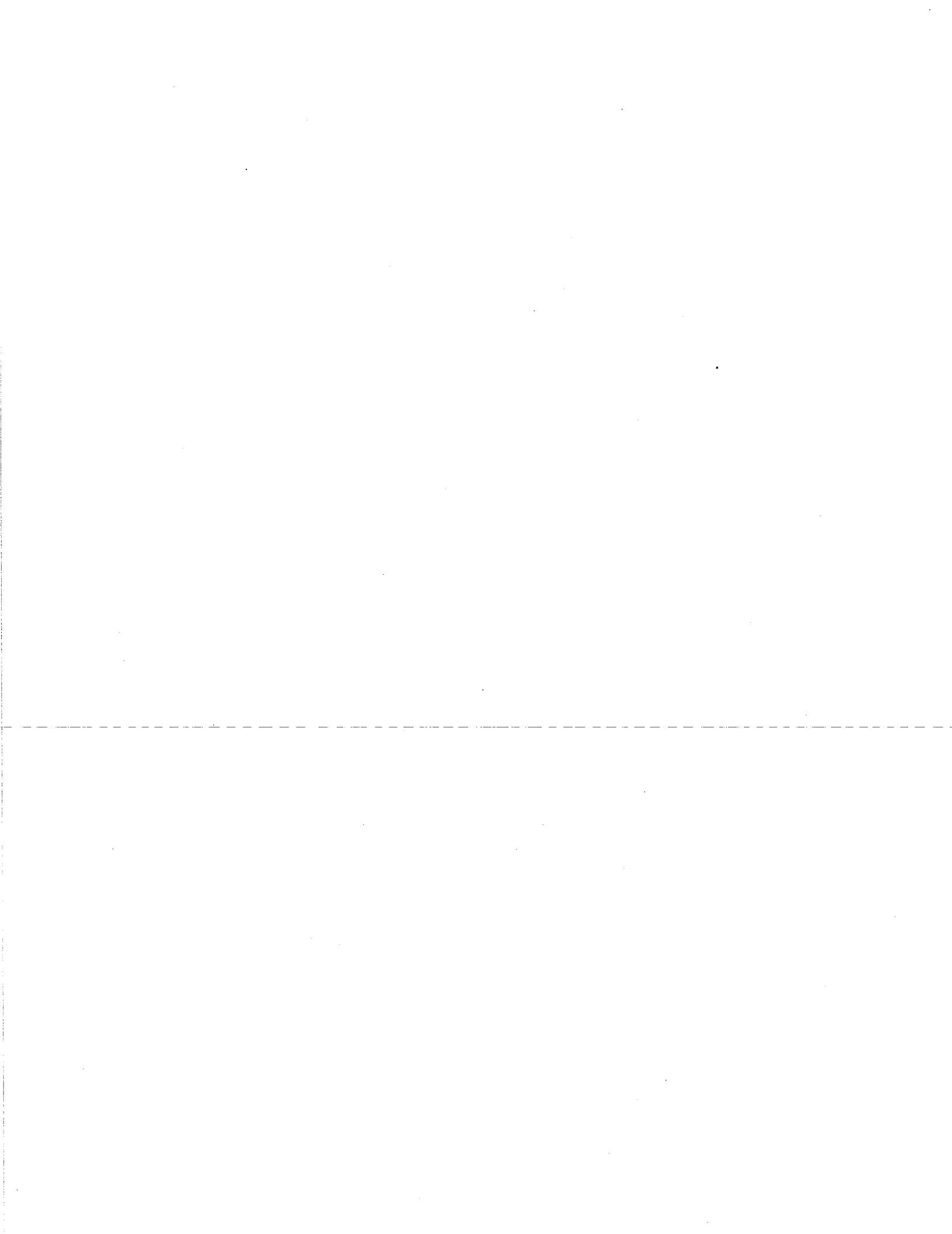
- 75) If prices are rising at rate of $0.5\%/\text{month}$, then they are rising at $6\%/\text{yr.}$ \rightarrow TRUE

Use unit conversion:

$$\frac{0.5\%}{\text{month}} \cdot \frac{12 \text{ month}}{\text{year}} = 6\%/\text{year}$$

- 76) The differential equation modeling exponential growth is $\frac{dy}{dx} = ky$ where k is constant \rightarrow TRUE

$$\hookrightarrow \int \frac{dy}{y} = \int k dx \quad (\ln|y|) = kx + c \quad \boxed{y = Ce^{kx}}$$



6.2 Homework (continued)

71) Newton's law of cooling: the rate of change in temperature of an object is proportional to the difference between the object's temperature and temperature of the surrounding medium.

When object is removed from furnace and placed in constant (80°F) ^{medium temp.}
the core temperature is 1500°F . One hour later the
core temperature is 1120°F . Find the core temperature 5 hrs. later.
(t , time; y , temperature)

$$(0, 1500)$$

$$(1, 1120)$$

$$(5, \underline{\hspace{2cm}})$$

$$\frac{dy}{dt} = k(y - 80)$$

$$\int \frac{dy}{y-80} = \int k dt$$

$$u = y - 80$$

$$\frac{du}{dy} = 1 \quad du = dy$$

$$\int \frac{du}{u} = \int k dt$$

$$\ln|y-80| = kt + C$$

$$e^{kt+C} = e^C e^{kt}$$

$$y - 80 = e^{kt} \cdot e^C$$

$$y - 80 = e^{kt} \cdot C$$

$$y - 80 = Ce^{kt}$$

$$1500 - 80 = Ce^{k(0)}$$

$$1420 = C$$

$$y - 80 = 1420e^{kt}$$

$$1120 - 80 = 1420e^{k(1)}$$

$$1040 = 1420e^k$$

$$0.7324 = e^k$$

$$\ln 0.7324 = \ln e^k$$

$$\ln 0.7324 = k$$

$$y - 80 = 1420e^{\ln 0.7324 t}$$

* plug in $t = 5$ and solve for y .

$$y - 80 = 1420e^{\ln 0.7324(5)}$$

$$y - 80 = 299.247$$

$$y = 379.247^{\circ}$$

6.2 Homework (continued)

35) Radium has a half-life of 1,599 yrs. Given that after 10,000 yrs only 0.5g remain, find the initial quantity and amount after 1000 yrs.

$$(0, C)$$

$$(1599, \frac{1}{2}C)$$

$$y = Ce^{kt}$$

solve for k first

$$\frac{1}{2}C = Ce^{1599k}$$

$$\frac{1}{2} = e^{1599k}$$

$$\ln 0.5 = \ln e$$

$$\ln 0.5 = 1599k$$

$$\frac{\ln 0.5}{1599} = k$$

$$(10,000, 0.5)$$

$$y = Ce^{\frac{\ln 0.5}{1599}t}$$

← plug in to solve for C

$$0.5 = Ce^{\frac{\ln 0.5}{1599}(10,000)}$$

$$0.5 = C(0.13103)$$

$$\frac{0.5}{0.13103} = C$$

$$C = 38.158$$

$$y = 38.158e^{\frac{\ln 0.5}{1599}t}$$

$$y = 38.158e^{\frac{\ln 0.5}{1599}t}$$

* plug in $t=1000$ to solve for y

$$y = 38.158e^{\frac{\ln 0.5}{1599}(1000)}$$

$$y = 24.736 \text{ g}$$

57) Find growth model for Bulgaria and let $t=0$ represent year 2000
(time t yrs, population in millions)

$$(1, 7.7)$$

$$K = -0.009$$

$$P = Ce^{kt}$$

$$7.7 = Ce^{-0.009(1)}$$

$$7.7 = C e^{-0.009(1)}$$

$$7.7 = C e^{-0.009(1)}$$

$$\frac{7.7}{e^{-0.009}} = C$$

$$C = 7.7696$$

$$a) P = 7.7696e^{-0.009t}$$

b) Use model to predict population in 2015
plug in $t=15$, solve for P.

$$P = 7.7696e^{-0.009(15)}$$

$$P = 6.788 \text{ million}$$

c) Since $K < 0$, the population is decreasing.

6.2 Homework p. 418-419 #17-25 odd, 33, 35, 57, 59, 63, 71,
 Differential Equation Word Problems
 (Exponential Growth/Decay)

73-76 all

- 21) Rate of change of y is proportional to y . When $x=0, y=4$
 and when $x=3, y=10$. Find the value of y when $x=6$.

$y' = ky$	$(0, 4)$	$\int \frac{dy}{y} = \int kdx$	$4 = Ce^{k(0)}$ $4 = C$ $y = 4e^{kx}$ $10 = 4e^{k(3)}$ $\frac{10}{4} = e^{3k}$ $2.5 = e^{3k}$	$\ln 2.5 = \ln e^{3k}$ $\ln 2.5 = 3k$ $\frac{\ln 2.5}{3} = k$ $y = 4e^{\frac{\ln 2.5}{3}x}$ $y = 4e^{\frac{\ln 2.5}{3}(6)}$ $y = 25$
$\frac{dy}{dx} = ky$	$(3, 10)$	$\ln y = kx + C$		
$\frac{dy}{y} = kdx$	$(6, \underline{\hspace{2cm}})$	$y = e^{kx} \cdot e^C$ $y = Ce^{kx}$		

- 23) Rate of change of V is proportional to V . When $t=0, V=20,000$ and
 when $t=4, V=12,500$. Find value of V when $t=6$.

(t, V)	$\ln V = kt + C$	$V = 20000e^{\frac{\ln 0.625}{4}t}$
$(0, 20,000)$	$e^{\underline{\hspace{2cm}}} \cdot e^{\underline{\hspace{2cm}}}$	* plug in $t=6$ and solve for V
$(4, 12,500)$	$V = e^{kt} \cdot e^C$	
$(6, \underline{\hspace{2cm}})$	$V = Ce^{kt}$ $20,000 = Ce^0$ $V = 20,000e^{kt}$	$V = 20,000e^{\frac{\ln 0.625}{4}(6)}$
$V' = KV$	$V = 20,000e^{kt}$	$V = 9882.118$
$\frac{dV}{dt} = KV$	$12,500 = 20,000e^{4k}$ $0.625 = e^{4k}$	
$\int \frac{dv}{v} = \int kdt$	$\ln 0.625 = \ln e^{4k}$ $\ln 0.625 = 4k$ $\frac{\ln 0.625}{4} = k$	

6'

Example 4: In a certain culture where the rate of growth of bacteria is proportional to the amount present, the number triples in 3 hours. If at the end of 12 hours there were 10 million bacteria, how many were present initially? *write specific equation:*

(time, bacteria)

(0, C)

(3, 3C)

(12, 10)

$$b' = kb$$

$$\frac{db}{dt} = kb$$

$$(1) \frac{db}{b} = k dt$$

$$\int \frac{db}{b} = \int k dt$$

$$\ln|b| = kt + C$$

$$\ln|b| = kt + C$$

$$e^{\ln|b|} = e^{kt+C}$$

$$|b| = e^{kt} \cdot e^C$$

$$b = e^{kt} \cdot C$$

$$b = Ce^{kt}$$

* solve for k first

$$b = Ce^{kt}$$

$$3b = Ce^{k(3)}$$

$$3 = e^{3k}$$

$$\ln 3 = \ln e^{3k}$$

$$\ln 3 = 3k$$

$$\frac{\ln 3}{3} = k$$

$$b = Ce^{\left(\frac{1}{3}\ln 3\right)t}$$

* Now solve for C
using (12, 10)

$$10 = Ce^{\left(\frac{1}{3}\ln 3\right)(12)}$$

$$10 = Ce^{4\ln 3}$$

$$\frac{10}{e^{4\ln 3}} = C$$

$$C \approx 0.123 \text{ million bacteria}$$

$$\text{or } 123,000 \text{ bacteria}$$

$$b) \text{ since } C = \frac{10}{e^{4\ln 3}} = 10e^{-4\ln 3}$$

$$b = 10e^{-4\ln 3} \cdot e^{\frac{1}{3}\ln 3 t}$$

$$b = 10e^{-4\ln 3 + \left(\frac{1}{3}\ln 3\right)t}$$

6.2 Notes (continued)

Ex. 3 The rate of decay of radium is proportional to the amount present at any time. If 60 mg of radium are present now and its half-life is 1690 years, how much radium will be present 100 years from now?

(t , r)
(year, radium amount)

$$(0, 60)$$

$$(100, -)$$

$$(1690, 30)$$

$$r' = kr$$

$$\frac{dr}{dt} = kr$$

$$\int \frac{dr}{r} = \int k dt$$

$$\ln|r| = kt + C$$

$$e^{\ln|r|} = e^{kt+C}$$

$$r = e^{kt} \cdot e^C$$

$$r = Ce^{kt}$$

$$r = Ce^{kt}$$

* solve for C
using (0, 60)

$$60 = Ce^{k(0)}$$

$$\frac{60}{60} = C e^{kt}$$

* solve for k
using (1690, 30)

$$30 = 60e^{k(1690)}$$

$$\frac{30}{60} = e^{1690k}$$

$$0.5 = e^{1690k}$$

$$\ln(0.5) = \ln e^{1690k}$$

$$\ln(0.5) = 1690k$$

$$\frac{\ln 0.5}{1690} = k$$

$$r = 60e^{\frac{\ln 0.5}{1690} t}$$

* plug in $t=100$ and solve for r

$$r = 60e^{\frac{\ln 0.5}{1690}(100)}$$

$$r = 57.589 \text{ mg}$$

6.2 Notes Differential Equations Word Problems (Exponential Growth/Decay)

- 1) Direct proportion equation: $y = kx$ * k is called the constant of proportionality
- 2) Inverse (Indirect) proportion equation: $y = \frac{k}{x}$

Ex.1 If the rate of change of y varies directly with the value of y , find the general equation.

$$\begin{aligned} y' &= ky & \int \frac{dy}{y} &= \int k dt \\ \frac{dy}{dt} &= ky & \ln|y| &= kt + C \end{aligned}$$

$$y = e^{kt+C} = e^{kt} \cdot e^C$$

$$y = C e^{kt}$$

This form will come up frequently. You may be familiar with $A = Pe^{rt}$ compounding interest formula

Ex.2 The rate of increase of the population of a certain city is proportional to the population. If the population in 1930 was 50,000 and in 1960 it was 75,000, find the expected population in 1990.

Let $t=0$ represent 1930.
Therefore $t=30$ for 1960
and $t=60$ for 1990

$$P' = KP$$

$$\frac{dP}{dt} = KP$$

$$\int \frac{dP}{P} = \int K dt$$

$$\ln|P| = kt + C$$

$$\ln|P| = kt + C$$

$$P = e^{kt+C}$$

$$P = e^{kt} \cdot C$$

$$P = Ce^{kt}$$

* Use given information to solve for C , then K .

$$(0, 50,000)$$

$$(30, 75,000)$$

$$(60, \underline{\hspace{2cm}})$$

$$\begin{aligned} 50,000 &= Ce^{k(0)} & 50,000 &= C(1) \\ P &= 50,000e^{kt} & \leftarrow & \text{Now solve for } K \text{ using } (30, 75,000) \\ 75,000 &= 50,000e^{k(30)} & 1.5 &= e^{30k} \\ 1.5 &= e^{30k} & \ln(1.5) &= \ln e^{30k} \\ \ln(1.5) &= 30k & \ln(1.5) &= 30k \\ \frac{\ln(1.5)}{30} &= k & P &= 50,000e^{\frac{\ln(1.5)}{30}t} \\ & & \leftarrow & \text{* Plug in } t=60 \text{ to find population.} \\ & & P &= 50,000e^{\frac{\ln(1.5)}{30}(60)} \end{aligned}$$

$$P = 112,500$$

- 3) The rate of decay of radium is proportional to the amount present at any time. If 60 mg of radium are present now and its half-life is 1690 years, how much radium will be present 100 years from now?
4. In a certain culture where the rate of growth of bacteria is proportional to the amount present, the number triples in 3 hours.
- A) If at the end of 12 hours there were 10 million bacteria, how many were present initially?
- B) Find the specific exponential growth equation

Note for homework: Newton's law of cooling: the rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding medium

1. Direct proportion equation :
2. Inverse (indirect) proportion equation:
3. k is called the _____

Exponential Growth/Decay class examples

1) If the rate of change of y varies directly with the value of y , find the general equation:

2) The rate of increase of the population of a certain city is proportional to the population. If the population in 1930 was 50,000 and in 1960 it was 75,000, what was the expected population in 1990?

1. Direct proportion equation: $y = kx$

2. Inverse (indirect) proportion equation: $y = \frac{k}{x}$

3. k is called the constant of proportionality

Exponential Growth/Decay class examples

1) If the rate of change of y varies directly with the value of y , find the general equation.

$$\begin{array}{c|c|c|c} y' = ky & \frac{dy}{y} = k dt & \int \frac{1}{y} dy = k \int dt & y = Ce^{kt} \\ \hline \frac{dy}{dt} = ky & \int \frac{dy}{y} = \int k dt & e^{\ln|y|} = e^{kt+C} & \\ & & |y| = e^{kt} \cdot e^C & \end{array}$$

2) The rate of increase of the population of a certain city is proportional to the population. If the population in 1930 was 50,000 and in 1960 it was 75,000, what was the expected population in 1990?

$P' = kP$ $P = Ce^{kt}$ $50,000 = Ce^{k(0)}$ $\underline{50,000 = C}$ $P = 50,000e^{kt}$ $\frac{75,000}{50,000} = \frac{50,000e^{k(36)}}{50,000}$	$(\text{time, Population})$ $(0, 50,000)$ $(30, 75,000)$ $(60, \underline{\hspace{2cm}})$	$\text{let } 1930 \rightarrow t = 0$ $P = 50,000e^{\left(\frac{\ln 1.5}{30}\right)t}$ $\underline{P = 50,000e^{\left(\frac{\ln 1.5}{30}\right)(60)}}$ $P = 112,500$ is the population in 1990
$1.5 = e^{30k}$ $\ln 1.5 = \ln e^{30k}$ $\ln 1.5 = \frac{30k}{30} \ln e$		$K = \frac{\ln 1.5}{30}$

$$r' = kr$$

- 3) The rate of decay of radium is proportional to the amount present at any time. If 60 mg of radium are present now and its half-life is 1690 years, how much radium will be present 100 years from now?

$$\begin{aligned}
 r &= Ce^{kt} && (\text{time, radium}) && \text{let now be } t=0 \\
 60 &= Ce^{k(0)} && (0, 60) && \boxed{\frac{\ln 0.5}{1690} = k} \\
 \underline{60 = C} & && (1690, 30) && r = 60e^{\left(\frac{\ln 0.5}{1690}\right)t} \\
 r &= 60e^{kt} && (100, \underline{\hspace{2cm}}) && r = 60e^{\left(\frac{\ln 0.5}{1690}\right)(100)} \\
 \frac{30}{60} &= \frac{60e^{k(1690)}}{60} && \ln 0.5 = \ln e^{1690k} && \boxed{r = 57.589 \text{ mg}}
 \end{aligned}$$

4. In a certain culture where the rate of growth of bacteria is proportional to the amount present, the number triples in 3 hours.

- A) If at the end of 12 hours there were 10 million bacteria, how many were present initially?
 B) Find the specific exponential growth equation

Note for homework: Newton's law of cooling: the rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding medium

Ch.6.2 Differential Equation Word Problem p.412-414

#17-33 odd, 37

$$17) \frac{dy}{dt} = -\frac{1}{2}y$$

$$\int \frac{dy}{y} = \int -\frac{1}{2} dt$$

passes through point (0, 10)

$$\ln|y| = -\frac{1}{2}t + C \quad |y| = e^{-\frac{1}{2}t} \cdot e^C \quad |y| = Ce^{-\frac{1}{2}t}$$

$$e^{\ln|y|} = e^{-\frac{1}{2}t + C} \quad |y| = e^{-\frac{1}{2}t} \cdot C \quad y = Ce^{-\frac{1}{2}t}$$

$$10 = Ce^{-\frac{1}{2}(0)}$$

$$10 = C$$

$$y = 10e^{-\frac{1}{2}t}$$

$$19) N' = KN$$

$$\frac{dN}{dt} = KN$$

$$\int \frac{dN}{N} = \int K dt$$

(time, N)

$$(0, 250)$$

$$(1, 400)$$

$$(4, -)$$

$$N = Ce^{kt}$$

\Rightarrow

$$250 = Ce^{k(0)}$$

$$N = 250e^{kt}$$

$$400 = 250e^{k(1)}$$

$$\frac{8}{5} = e^k$$

$$C = 250$$

$$\ln(\frac{8}{5}) = \ln e^k$$

$$\ln(\frac{8}{5}) = k$$

$$N = 250e^{\ln(\frac{8}{5})t}$$

$$N = 250e^{\ln(\frac{8}{5})(4)}$$

$$N \approx 1638.4 = \frac{8192}{5}$$

23) Find a specific equation in the form $y = Ce^{kt}$ passing through (1, 5) and (5, 2)

$$\text{point } (1, 5) \rightarrow 5 = Ce^{k(1)} \rightarrow 10 = 2Ce^k$$

$$\text{point } (5, 2) \rightarrow 2 = Ce^{k(5)} \rightarrow 10 = 5Ce^{5k}$$

$$2Ce^k = 5Ce^{5k}$$

$$\frac{2}{5} = \frac{e^{5k}}{e^k}$$

$$\frac{2}{5} = e^{4k}$$

$$\ln(\frac{2}{5}) = \ln e^{4k}$$

$$\ln(\frac{2}{5}) = 4k$$

$$k = \frac{1}{4}\ln(\frac{2}{5})$$

$$5 = Ce^k$$

$$5 = Ce^{\frac{1}{4}\ln(\frac{2}{5})}$$

$$\frac{5}{e^{\frac{1}{4}\ln(\frac{2}{5})}} = C$$

$$C \approx 6.287$$

$$y = 6.287e^{\frac{1}{4}\ln(\frac{2}{5})t}$$

29) $y = Ce^{kt}$ half-life: 1599 yrs.
 initial: 20g (time, Amount)

$$20 = Ce^{k(0)} \rightarrow C = 20$$

$$y = 20e^{kt}$$

$$10 = 20e^{k(1599)}$$

$$\frac{10}{20} = e^{1599k}$$

$$\frac{1}{2} = e^{1599k}$$

$$\begin{cases} \ln\left(\frac{1}{2}\right) = \ln e^{1599k} \\ \ln\left(\frac{1}{2}\right) = 1599k \\ \frac{\ln\left(\frac{1}{2}\right)}{1599} = k \end{cases} \quad \begin{cases} y = 20e^{\frac{\ln\left(\frac{1}{2}\right)}{1599}t} \\ y = 20e^{\frac{\ln\left(\frac{1}{2}\right)}{1599}(1000)} \\ y = 20e^{\frac{\ln\left(\frac{1}{2}\right)}{1599}(10000)} \end{cases}$$

1000 yrs.

10,000 yrs.

$$(1599, 10)$$

$$(1000, -)$$

$$(10,000, -)$$

37) $y = Ce^{kt}$ half-life: 1599 yrs. (time, Amount)

$$y = Ce^{kt}$$

$$\frac{1}{2}R = Re^{1599t}$$

$$\frac{1}{2} = e^{1599t}$$

$$\ln\left(\frac{1}{2}\right) = \ln e^{1599t}$$

$$\ln\left(\frac{1}{2}\right) = 1599t$$

$$\frac{\ln\left(\frac{1}{2}\right)}{1599} = t$$

$$\begin{cases} y = Ce^{\frac{\ln\left(\frac{1}{2}\right)}{1599}t} \\ y = Ce^{\frac{\ln\left(\frac{1}{2}\right)}{1599}(100)} \\ y = C(0.9575) \end{cases}$$

Therefore, 95.76% remains after 100 yrs.

Differential Equations Unit

Slope Fields (Direction Fields) Notes

Slope Fields: a graphical approach to look at all the solutions of a differential equation. Slope fields consists of short line segments representing slope (steepness) sketched at lots of different points

It consists of a set of short line segments drawn on a pair of axes. These line segments are the tangents to a family of solution curves for the differential equation at various points. The tangents show the direction in which the solution curves will follow. Slope fields are useful in sketching solution curves without having to solve a differential equation algebraically.

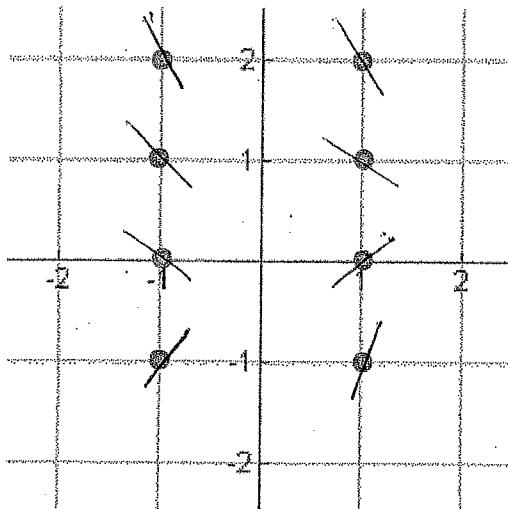
Steps:

- 1) Identify the ordered pairs indicated on the graph.
- 2) Plug in the ordered pairs in the differential equation to find slope
- 3) Sketch a short line segment representing the slope through the given point
- 4) Repeat this for all remaining ordered pairs.

Example 1: Sketch a slope field for the given differential equation at the indicated eight points.

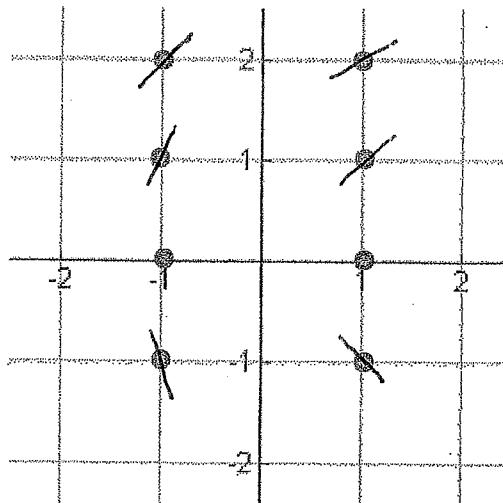
a) $\frac{dy}{dx} = x - 2y$

x	-1	-1	-1	-1	1	1	1	1
y	2	1	0	-1	2	1	0	-1
y'	-5	-3	-1	1	-3	-1	1	3



b) $\frac{dy}{dx} = \frac{2-x}{y}$

x	-1	-1	-1	-1	1	1	1	1
y	2	1	0	-1	2	1	0	-1
y'	$\frac{3}{2}$	3	und	-3	$\frac{1}{2}$	1	und.	-1



Determine the differential equation being graphed by each of the slope fields below. Then sketch a solution that passes through the indicated point.

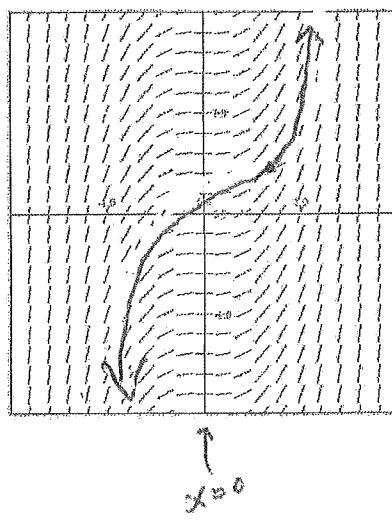
1. a) $\frac{dy}{dx} = x^3$

b) $\frac{dy}{dx} = 3x^2$ All slopes are positive except y-axis

c) $\frac{dy}{dx} = 2x + y$

d) $\frac{dy}{dx} = \frac{x}{y}$

e) $\frac{dy}{dx} = \ln x$



$$\frac{dy}{dx} = 3x^2 \quad \int dy = \int 3x^2 dx$$

$$y = \frac{3x^3}{3} + C$$

$$y = x^3 + C$$

*The shape of the slope field graph resembles the family of functions of the general solution above, $y = x^3 + C$

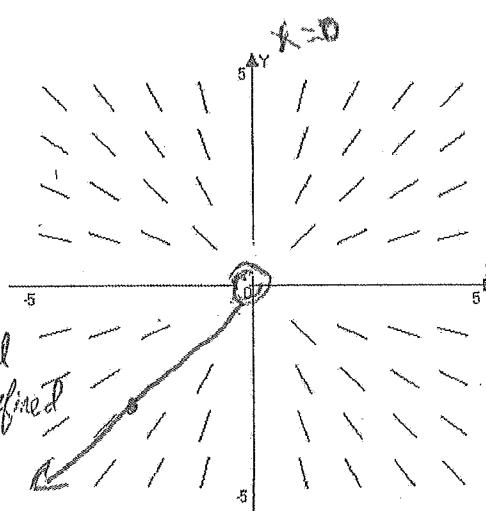
3. a) $\frac{dy}{dx} = x - 2$

b) $\frac{dy}{dx} = x^3$

c) $\frac{dy}{dx} = x - y$

d) $\frac{dy}{dx} = \frac{y}{x}$ y-axis ordered pairs are undefined and x-axis has slope = 0

e) $\frac{dy}{dx} = e^y$



$$\frac{dy}{dx} = \frac{y}{x} \quad \int \frac{dy}{y} = \int \frac{dx}{x}$$

$$e^{\ln y} = e^{\ln|x| + C} = e^{\ln x} \cdot e^C$$

$$y = Cx$$

*The slope field resembles the linear equation of the general solution $y = Cx$

Sketch slope fields for the following differential equation. Then find the general solution analytically

4. $\frac{dy}{dx} = \frac{-x}{y}$

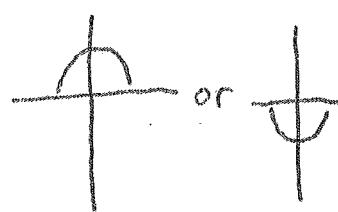
$$\int y dy = \int -x dx$$

$$y = \pm \sqrt{-x^2 + C_1}$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + C$$

$$y^2 = 2\left(\frac{-x^2}{2}\right) + 2C$$

$$y^2 = -x^2 + 2C$$



Slope Fields Homework

Sketch slope fields for each of the following differential equations. Then find the general solution analytically.
For problems 4 through 6, find the general solution first, then the specific solution that passes through the given point and state the domain of that solution

$$1. \frac{dy}{dx} = 4 - y$$

$$\int \frac{dy}{4-y} = \int dx$$

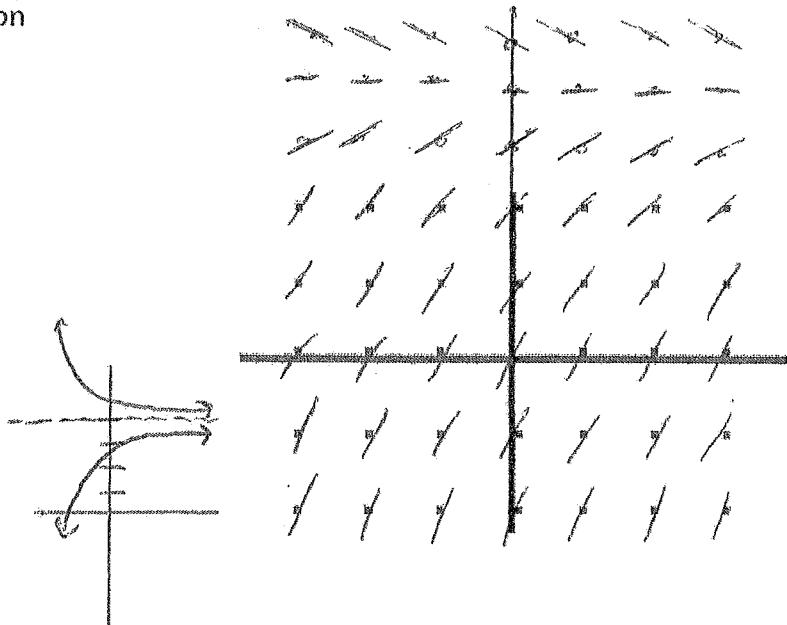
$$u = 4-y$$

$$\frac{du}{dy} = -1$$

$$dy = -1 du$$

$$\int -\frac{du}{u} = \int dx$$

$$\begin{aligned} -\ln|u| &= x + C \\ -\ln|4-y| &= x + C \\ \ln|4-y| &= -x - C \\ e^{\ln|4-y|} &= e^{-x-C} \\ 4-y &= Ce^{-x} \\ -y &= Ce^{-x} - 4 \\ y &= Ce^{-x} + 4 \end{aligned}$$

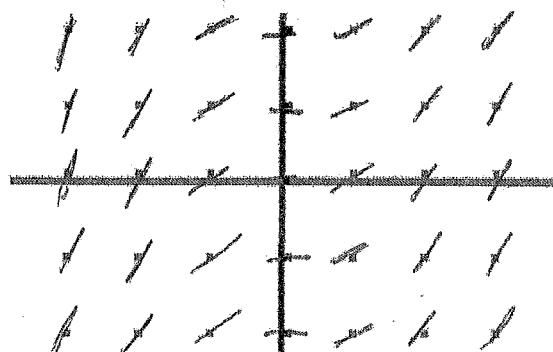
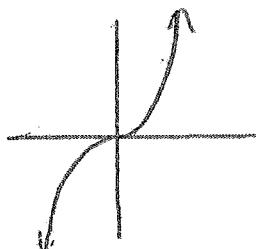


$$2. \frac{dy}{dx} = 3x^2$$

$$\int dy = \int 3x^2 dx$$

$$y = \frac{3x^3}{3} + C$$

$$y = x^3 + C$$



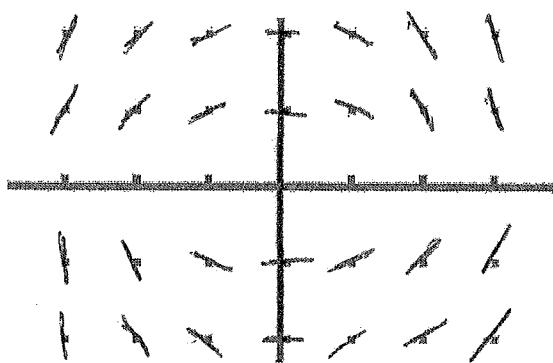
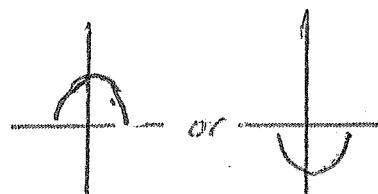
$$3. \frac{dy}{dx} = \frac{-x}{y}$$

$$\int y dy = \int -x dx \quad y = \pm \sqrt{-x^2 + C_1}$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + C_1$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + C_1$$

$$y^2 = -x^2 + 2C_1$$



$$4. \frac{dy}{dx} = \frac{1}{y} \quad \text{point } (1, 3)$$

$$\int y dy = \int dx$$

$$\frac{y^2}{2} = x + C$$

$$\frac{9}{2} = 1 + C$$

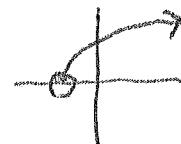
$$\frac{7}{2} = C$$

$$\frac{y^2}{2} = x + \frac{7}{2}$$

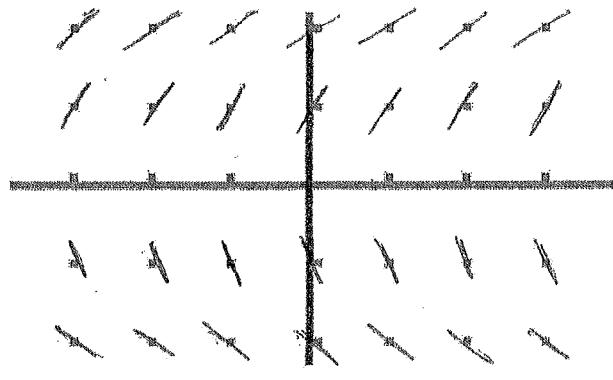
$$y^2 = 2x + 7$$

$$y = \pm \sqrt{2x+7}$$

$$y = \sqrt{2x+7}$$



$$\text{Domain: } (-\frac{7}{2}, \infty)$$



$$5. \frac{dy}{dx} = \frac{xy}{8} \quad \text{point } (0, -2)$$

$$\int \frac{dy}{y} = \int \frac{1}{8} x dx$$

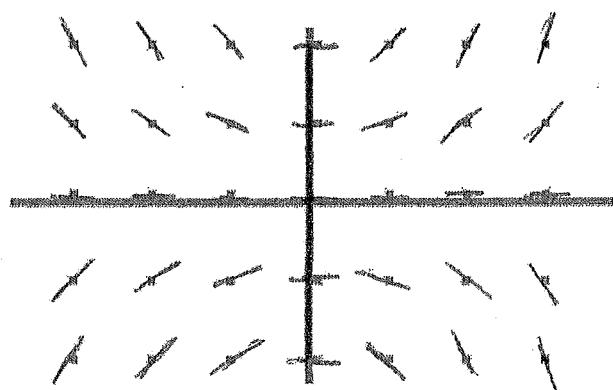
$$\begin{aligned} -2 &= Ce^{\frac{x^2}{16}} \\ y &= -2e^{\frac{x^2}{16}} \end{aligned}$$

$$\ln|y| = \frac{1}{8}\left(\frac{x^2}{2}\right) + C$$

$$e^{\ln|y|} = e^{\frac{x^2}{16} + C}$$

$$y = Ce^{\frac{x^2}{16}}$$

$$\text{Domain: } (-\infty, \infty)$$



$$6. \frac{dy}{dx} = \frac{y}{x} \quad \text{point } (3, -6)$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

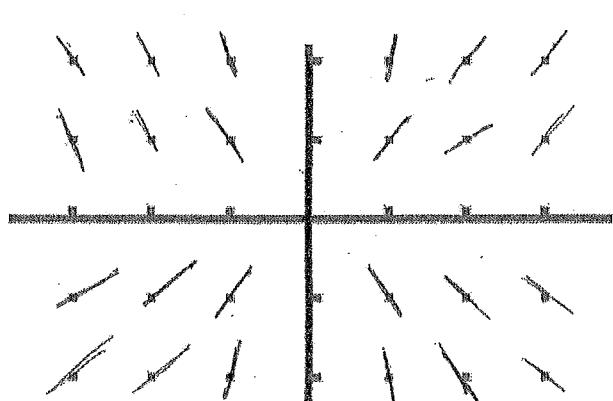
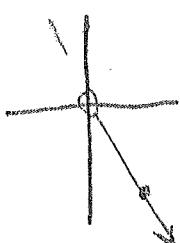
$$\begin{aligned} \ln|y| &= \ln|x| + C \\ e^{\ln|y|} &= e^{\ln|x|} \cdot e^C \end{aligned}$$

$$y = Ce^{\ln x}$$

$$-6 = C(3)$$

$$-2 = C$$

$$y = -2x$$



$$\text{Domain: } (0, \infty)$$

Slope Fields: Additional Notes

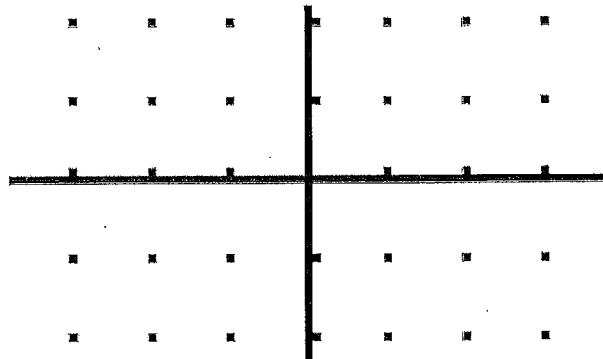
II. Finding Domain for Differential Equations

The domain of the solution to a differential equation is the largest open interval containing the initial value given for which both the differential equation and the solution are defined. This is called the “maximum interval of existence”

1. Solve the differential equation, and find the specific solution
2. Find the initial domain of the specific solution
3. Further restrict domain by:
 - a. Finding domain of differential equation
 - b. Be sure the initial condition is included in the domain
4. Graph solution and use slope fields to help determine/confirm domain
5. Incorporate restrictions to find the final domain

** Domain can only be ONE open interval that includes the initial value

1. Given $\frac{dy}{dx} = \frac{1}{x^2y}$, and $y(-1) = -2$
 - a) Find the particular solution
 - b) Find the domain.
 - c) Find the equation of tangent line through $(-1, -2)$
 - d) Use it to approximate $y(-1.1)$



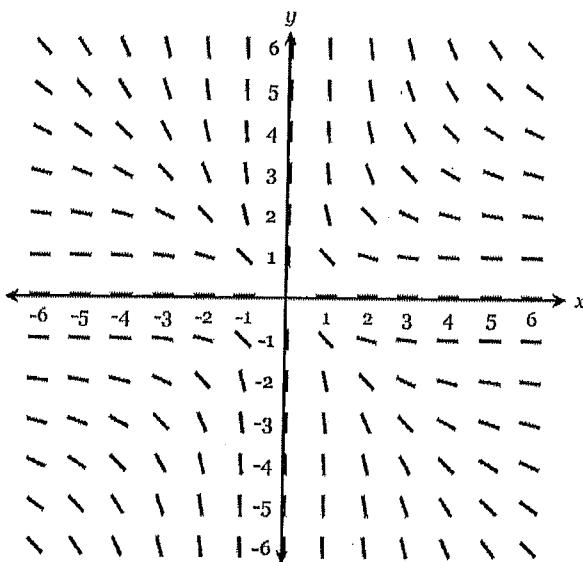
2. Let $P(t)$ represent the number of wolves in a population at time t in years, where $t \geq 0$. The population $P(t)$ is increasing at a rate directly proportional to $800 - P(t)$, where the constant of proportionality is k .
 - a) If $P(0) = 500$, find $P(t)$ in terms of t and k .
 - b) If $P(2) = 700$, find k .
 - c) Find $\lim_{t \rightarrow \infty} P(t)$

Slope Fields Practice WS

Key

Select the differential equation that matches the given slope field.

1)

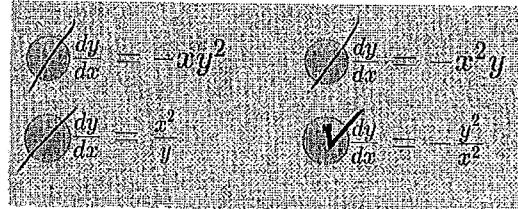


* when $x=0$, $\frac{dy}{dx}$ = undefined

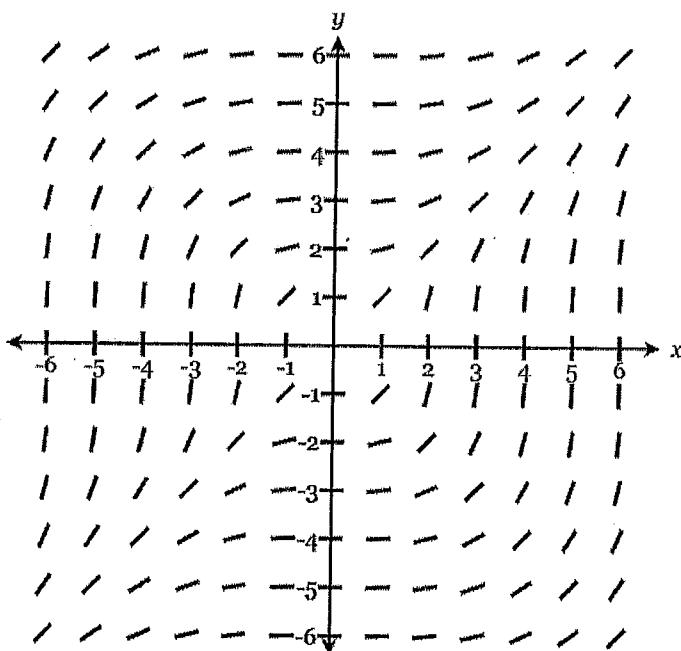
* when $y=0$, $\frac{dy}{dx} = 0$

* all other points displaying negative slopes

$$\frac{dy}{dx} = \frac{-y^2}{x^2}$$



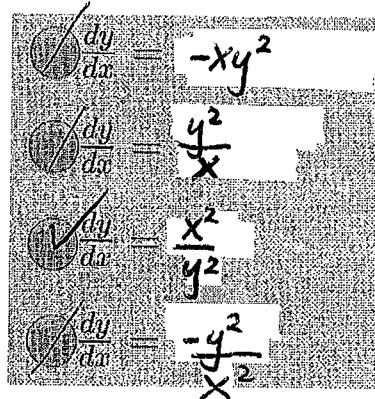
2)



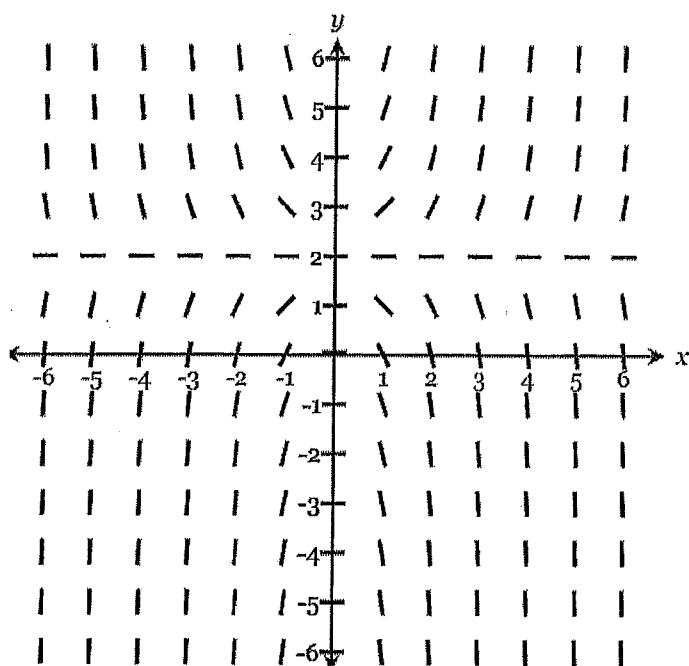
* when $x=y$, the slope is $\frac{dy}{dx} \approx 1$

* most ordered pairs showing positive slope

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$



3)



* when $y = 2$, $\frac{dy}{dx} = 0$

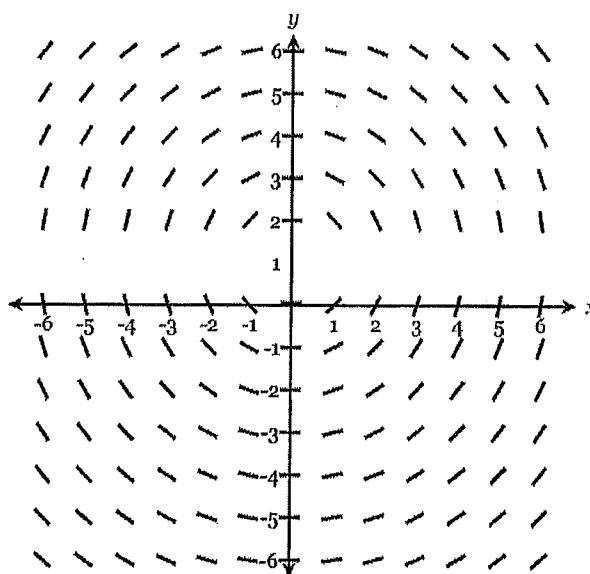
* when $x = 0$, $\frac{dy}{dx} = 0$

* when $x > 0$, and $y > 2$, $\frac{dy}{dx} > 0$

$$\boxed{\frac{dy}{dx} = x(y-2)}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{x^2}{(y-2)^2} \\ \frac{dy}{dx} &= -x^2(y-2)^2 \\ \frac{dy}{dx} &= x(y-2) \\ \frac{dy}{dx} &= \frac{y-2}{x}\end{aligned}$$

4)



* when $y = 1$, $\frac{dy}{dx}$ = undefined
 ex: $(-2, 3)$
 and $(2, -3)$

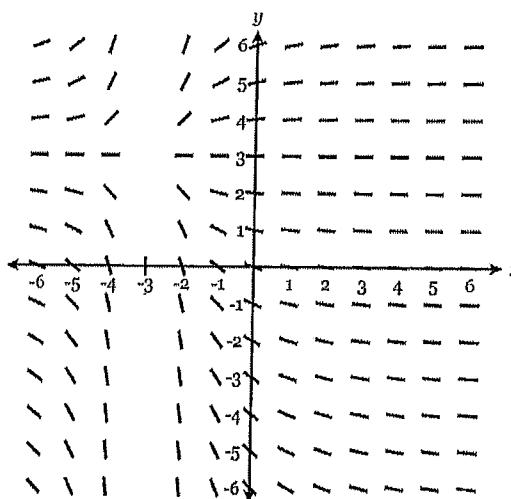
* most points in Q II, Q IV have $\frac{dy}{dx} > 0$

* most points in Q I, Q III have $\frac{dy}{dx}$
 (ex: $(2, 3)$ and $(-3, -2)$) $\frac{dy}{dx} < 0$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y-1}}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{x}{(y-1)^2} \\ \frac{dy}{dx} &= \frac{(y-1)^2}{x} \\ \frac{dy}{dx} &= \frac{x}{y-1} \\ \frac{dy}{dx} &= \frac{x^2}{y-1}\end{aligned}$$

5)



* when $x = -3$, $\frac{dy}{dx}$ = undefined

* when $y = 3$, $\frac{dy}{dx} = 0$

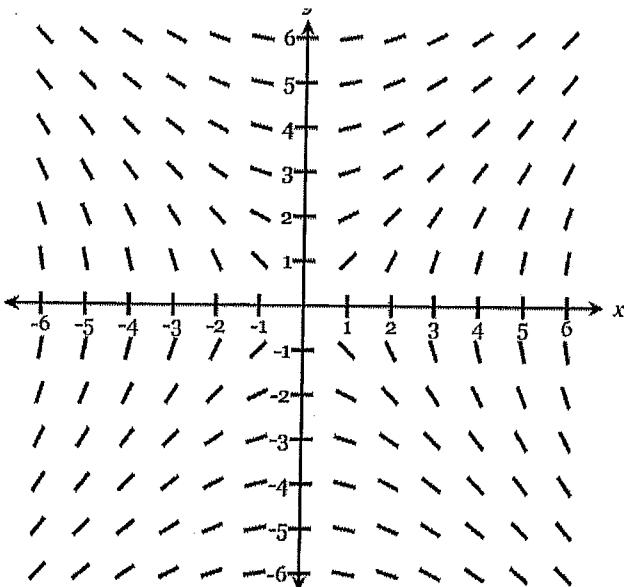
* when $y > 3$, $\frac{dy}{dx} > 0$

* when $y < 3$, $\frac{dy}{dx} < 0$

$$\begin{aligned}\cancel{\frac{dy}{dx}} &= -(x+3)(y-3) \\ \cancel{\frac{dy}{dx}} &= \frac{(y-3)^2}{(x+3)^2} \\ \cancel{\frac{dy}{dx}} &= \frac{x+3}{y-3} \\ \cancel{\frac{dy}{dx}} &= \frac{y-3}{(x+3)^2}\end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{y-3}{(x+3)^2}}$$

6)



* when $x = 0$, $\frac{dy}{dx} = 0$

* In Q1, Q3, $\frac{dy}{dx} > 0$

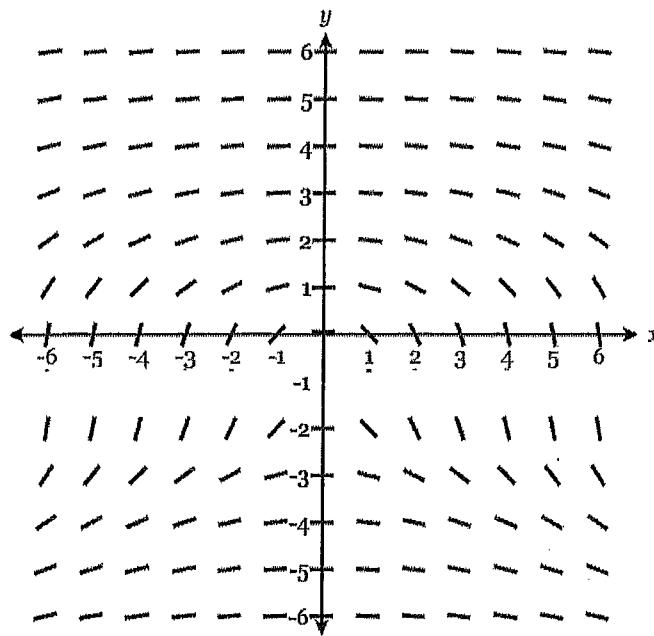
* In Q2, Q4, $\frac{dy}{dx} < 0$

* when $x = y$, $\frac{dy}{dx} = 1$

$$\boxed{\frac{dy}{dx} = \frac{x}{y}}$$

$$\begin{array}{ll}\cancel{\frac{dy}{dx}} = \frac{x}{y} & \cancel{\frac{dy}{dx}} = -xy \\ \cancel{\frac{dy}{dx}} = xy & \cancel{\frac{dy}{dx}} = -x^2y\end{array}$$

7)



* when $y = -1$, $\frac{dy}{dx}$ = undefined

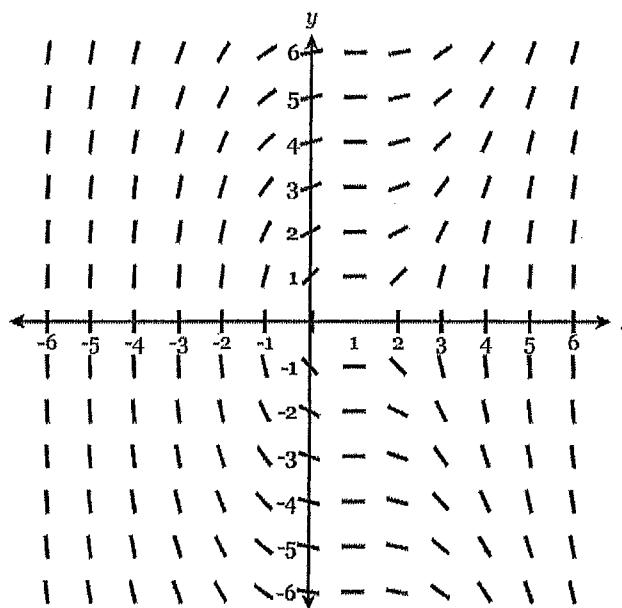
* when $x > 0$ most $\frac{dy}{dx} < 0$

* when $x < 0$, most $\frac{dy}{dx} > 0$

$$\boxed{\frac{dy}{dx} = \frac{-x}{(y+1)^2}}$$

$$\begin{aligned} \cancel{\frac{dy}{dx}} &= \cancel{-\frac{x}{(y+1)^2}} \\ \cancel{\frac{dy}{dx}} &= \cancel{x^2(y+1)^2} \\ \cancel{\frac{dy}{dx}} &= \cancel{(y+1)^2} \\ \cancel{\frac{dy}{dx}} &= \cancel{x^2} \\ \cancel{\frac{dy}{dx}} &= x^2(y+1) \end{aligned}$$

8)



* when $x = 1$, $\frac{dy}{dx} = 0$

* when $y > 0$, most $\frac{dy}{dx} > 0$

* when $y < 0$, most $\frac{dy}{dx} < 0$

$$\boxed{\frac{dy}{dx} = \frac{(x-1)^2}{y}}$$

$$\begin{aligned} \cancel{\frac{dy}{dx}} &= \cancel{\frac{(x-1)^2}{y}} \\ \cancel{\frac{dy}{dx}} &= \cancel{\frac{y}{(x-1)^2}} \\ \cancel{\frac{dy}{dx}} &= \cancel{\frac{y+1}{y^2}} \\ \cancel{\frac{dy}{dx}} &= \cancel{\frac{y}{(x-1)^2}} \end{aligned}$$

Gather general characteristics:

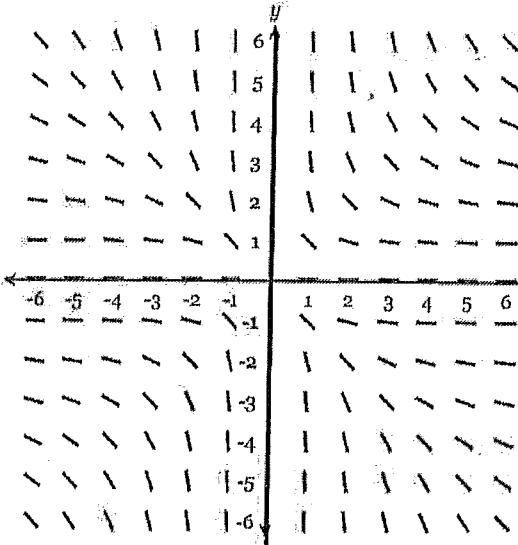
Slope Fields Practice WS

- a) slope = 0? b) slope undefined? c) pos. slope? d) neg. slope?

Select the differential equation that matches the given slope field.

1)

1) slope 0? ($y=0$)



2) slope und: ($x=0$)

3) + slope: no pos. slope

4) - slope: mostly negative slope

(1) $\frac{dy}{dx} = -xy^2$	(2) $\frac{dy}{dx} = -x^2y$	a) $\frac{dy}{dx} = -xy^2$
(3) $\frac{dy}{dx} = \frac{x^2}{y}$	(4) $\frac{dy}{dx} = \frac{y^2}{x}$	b) $\frac{dy}{dx} = -xy$

c) $\frac{dy}{dx} = \frac{x^2}{y}$

d) $\frac{dy}{dx} = \frac{-y^2}{x^2}$

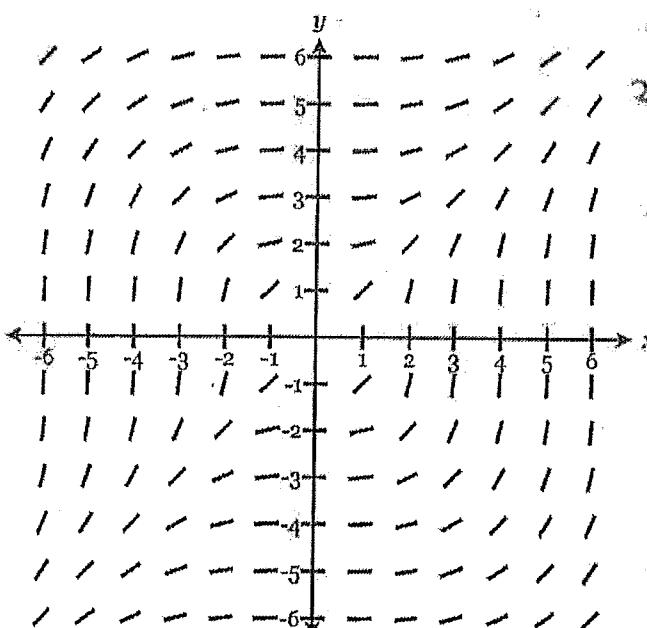
2)

1) slope = 0 : ($x=0$)

2) slope und: ($(0,0)$)

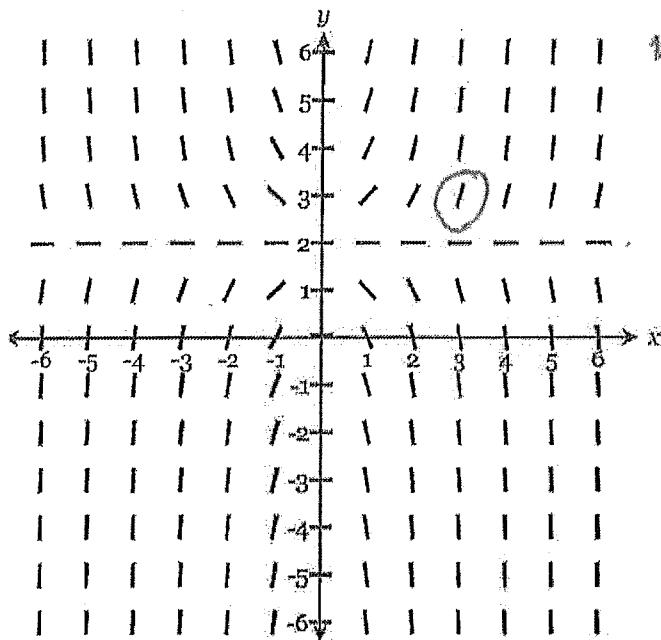
3) + slope (mostly positive)

4) - slope none



(1) $\frac{dy}{dx} = -xy^2$	(2) $\frac{dy}{dx} = \frac{y^2}{x}$	$\rightarrow (1,1)$
(3) $\frac{dy}{dx} = \frac{x^2}{y^2}$	(4) $\frac{dy}{dx} = \frac{-y^2}{x^2}$	

3)

1) slope = 0 ($y = 2$)

2) slope und (none)

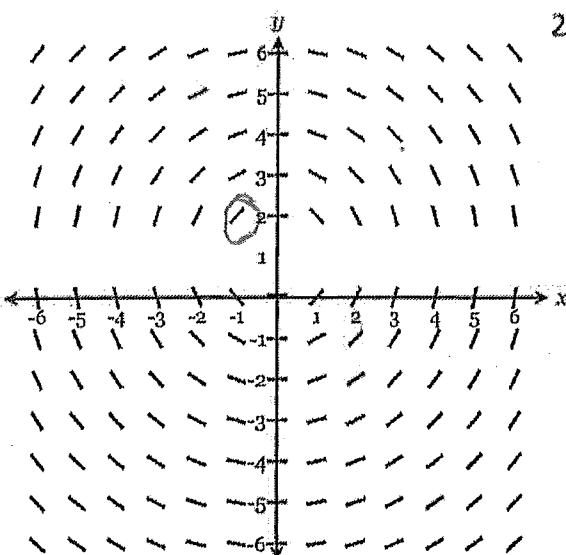
3) + slope most Q1 (above $y = 2$)
Q3

4) - slope Q2, Q4

$$\begin{aligned} \text{a)} & \frac{dy}{dx} = \frac{x^2}{(y-2)^2} \\ \text{b)} & \frac{dy}{dx} = -x^2(y-2)^2 \\ \text{c)} & \frac{dy}{dx} = x(y-2) \\ \text{d)} & \frac{dy}{dx} = \frac{y-2}{x} \end{aligned}$$

test
(3, 3)b) $-x^2(y-2)^2$ c) $x(y-2)$ d) $-y+2$

4)

1) slope = 0 (maybe $x=0$)? X2) slope und ($y = 1$)

3) + slope Q2, Q4

4) - slope Q1, Q3 (pos.)

test (-1, 2)

$$\begin{aligned} \text{a)} & \frac{dy}{dx} = \frac{-x}{(y-1)^2} \\ \text{b)} & \frac{dy}{dx} = -\frac{(y-1)^2}{x} \\ \text{c)} & \frac{dy}{dx} = \frac{-x}{y-1} \\ \text{d)} & \frac{dy}{dx} = \frac{x^2}{y-1} \end{aligned}$$

b) $-\frac{x}{(y-1)^2}$
(-2, -1)
(neg. slope)

(2, 2) < 0

d) $\frac{x^2}{y-1}$

1) $\frac{dy}{dx} = \frac{1}{x^2 y}$ $y(-1) = -2$

a) Find particular solution
c) Find tangent line equation

$\frac{x^2 y dy}{x^2} = \frac{1 dx}{x^2}$

$\int y dy = \int \frac{1}{x^2} dx$

$\int y dy = \int x^{-2} dx$

$2 \left(\frac{y^2}{2} = \frac{x^{-1}}{-1} + C \right)$

$y^2 = -2x^{-1} + C$

$y^2 = \frac{-2}{x} + C$

$(-2)^2 = \frac{-2}{-1} + C$

$4 = 2 + C$

$\underline{\underline{2 = C}}$

$y^2 = \frac{-2}{x} + 2$

$y^2 = \sqrt{\frac{-2}{x} + 2}$

$y = \sqrt{\frac{-2}{x} + 2}$

or

$y = -\sqrt{\frac{-2}{x} + 2}$

c) tangent line equation point: $(-1, -2)$

$$y - y_1 = m(x - x_1)$$

$$\text{slope: } \left. \frac{dy}{dx} \right|_{(-1, -2)} = \frac{1}{x^2 y} = \frac{1}{(-1)^2 (-2)} = -\frac{1}{2}$$

$$y - (-2) = -\frac{1}{2}(x - (-1))$$

$$\boxed{y + 2 = -\frac{1}{2}(x + 1)}$$

d) Approximate $y(-1.1)$

$$\begin{aligned} y &= -\frac{1}{2}(x+1) - 2 \\ y(-1.1) &= -\frac{1}{2}(-1.1+1) - 2 \end{aligned}$$

$$\boxed{y(-1.1) = -1.95}$$

Solving Differential Equations Task (Continued)

- 1) The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

$$\frac{dB}{dt} = \frac{100-B}{5}$$

$$\begin{cases} u = 100 - B \\ \frac{du}{dB} = -1 \end{cases}$$

$$5dB = (100 - B)dt$$

$$\frac{dB}{100-B} = \frac{dt}{5}$$

$$\int \frac{dB}{100-B} = \frac{1}{5} \int dt$$

$$\begin{cases} du = -dB \\ dB = -du \\ \int \frac{-1}{u} du \end{cases}$$

$$= -\ln|u|$$

$$100 - Ce^{-\frac{1}{5}t} = B$$

$$\underline{B = 100 - Ce^{-\frac{1}{5}t}} \quad (\text{general equation})$$

$$20 = 100 - Ce^{-\frac{1}{5}(0)}$$

$$20 = 100 - C$$

$$\underline{C = 80}$$

plug in $(0, 20)$

$$\boxed{B = 100 - 80e^{-\frac{1}{5}t}}$$

$$\boxed{B(t) = 100 - 80e^{-\frac{1}{5}t}}$$

t (time, Bird weight)
 B
 $(0, 20)$

$$\ln|100-B| = -\frac{1}{5}t + C$$

$$e^{\ln|100-B|} = e^{-\frac{1}{5}t + C}$$

$$|100-B| = e^{-\frac{1}{5}t} \cdot e^C$$

$$|100-B| = e^{-\frac{1}{5}t} \cdot C$$

$$|100-B| = Ce^{-\frac{1}{5}t}$$

$$100 - B = Ce^{-\frac{1}{5}t}$$

$$2) y' - xy\cos(x^2) = 0 \quad y(0) = e$$

$$\frac{dy}{dx} = xy\cos(x^2)$$

$$dy = xy\cos(x^2) dx$$

$$\frac{dy}{y} = x\cos(x^2) dx$$

$$\int \frac{1}{y} dy = \int x\cos(x^2) dx$$

$$\begin{aligned} u &= x^2 & dx &= \frac{du}{2x} \\ \frac{du}{dx} &= 2x \end{aligned}$$

$$\int x\cos u \cdot \frac{du}{2x}$$

$$\int \frac{1}{y} dy = \frac{1}{2} \int \cos u du$$

$$\ln|y| = \frac{1}{2} \sin u + C$$

$$\ln|y| = \frac{1}{2} \sin(x^2) + C$$

$$e^{\ln|y|} = e^{\frac{1}{2} \sin(x^2) + C}$$

$$|y| = e^{\frac{1}{2} \sin(x^2)} \cdot e^C$$

$$|y| = Ce^{\frac{1}{2} \sin(x^2)}$$

$$y = Ce^{\frac{1}{2} \sin(x^2)}$$

$$y = Ce^{\frac{1}{2} \sin(x^2)} \quad \text{plug in } y(0) = e$$

$$e = Ce^{\frac{1}{2} \sin(0)^2}$$

$$e = Ce^0$$

$$e = C$$

$$y = e \cdot e^{\frac{1}{2} \sin(x^2)}$$

$$y = e^{\frac{1}{2} \sin(x^2) + 1}$$

Solving Differential Equations: Additional Practice Problems

3) Solve the Differential Equation: $y \ln x^4 - xy' = 0$

$$y \ln x^4 - x \left(\frac{dy}{dx} \right) = 0$$

$$-x \frac{dy}{dx} = -y \ln x^4$$

$$-x dy = -y \ln x^4 dx$$

$$x dy = y \ln x^4 dx$$

$$\frac{dy}{y} = \frac{\ln x^4}{x} dx$$

$$\int \frac{1}{y} dy = \int \frac{4 \ln x}{x} dx \quad \begin{array}{l} u = \ln x \\ \frac{du}{dx} = \frac{1}{x} \end{array}$$

$$\downarrow \quad \begin{array}{l} 4 \int \frac{u}{x} \cdot x du \rightarrow 4 \int u du \\ 4 \left(\frac{u^2}{2} \right) \end{array}$$

$$\ln |y| = 2u^2 + C$$

$$\ln |y| = 2(\ln x)^2 + C$$

$$e^{\ln |y|} = e^{2(\ln x)^2 + C}$$

$$|y| = e^{2(\ln x)^2} \cdot e^C$$

$$|y| = e^{2(\ln x)^2} \cdot C$$

$$y = C e^{2(\ln x)^2}$$

4) a) Find the general solution

$$y\left(\frac{dy}{dx}\right) - 2e^{3x} = 0$$

$$\frac{ydy}{dx} = \frac{2e^{3x}}{1}$$

$$ydy = 2e^{3x} dx$$

b) Find the particular solution

$$yy' - 2e^{3x} = 0 \quad y(0) = 5$$

$$\int y dy = \int 2e^{3x} dx \quad \begin{array}{l} u = 3x \\ \frac{du}{dx} = 3 \end{array}$$

$$2 \int e^u \cdot \frac{du}{3} \quad dx = \frac{du}{3}$$

$$\int y dy = \frac{2}{3} \int e^u du$$

$$\frac{y^2}{2} = \frac{2}{3} e^{3x} + C \quad \begin{array}{l} \text{solve for } C: \\ y(0) = 5 \end{array}$$

$$\frac{y^2}{2} = \frac{2}{3} e^{3x} + \frac{71}{6}$$

$$\frac{5^2}{2} = \frac{2}{3} e^{3(0)} + C$$

$$2\left(\frac{y^2}{2} = \frac{2}{3} e^{3x} + \frac{71}{6}\right)$$

$$\frac{25}{2} = \frac{2}{3}(1) + C$$

$$y^2 = \frac{4}{3} e^{3x} + \frac{142}{6}$$

$$\frac{25}{2} - \frac{2}{3} = C$$

$$y = \pm \sqrt{\frac{4}{3} e^{3x} + \frac{71}{3}}$$

$$\frac{75}{6} - \frac{4}{6} = C$$

$$\frac{71}{6} = C$$

$$y = \sqrt{\frac{4}{3} e^{3x} + \frac{71}{3}} \quad \text{since } y(0) = 5$$

2nd

Key

Solving Differential Equations Task (part 2)

1)

The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

$$\frac{dB}{dt} = \frac{1}{5}(100 - B)$$

$$\frac{dB}{dt} = \frac{(100 - B)}{5}$$

$$\frac{5dB}{5} = \frac{(100 - B)dt}{100 - B}$$

$$\int \frac{dB}{100 - B} = \int \frac{dt}{5} \rightarrow \frac{1}{5} \int 1 dt.$$

$$u = 100 - B$$

$$\frac{du}{dB} = -1$$

$$dB = -1 du$$

$$\int \frac{-1 du}{u}$$

$$-\ln|u| \rightarrow -\ln|100 - B|$$

$$(-\ln|100 - B|) = \frac{1}{5}t + C$$

$$\ln|100 - B| = -\frac{1}{5}t + C$$

$$|100 - B| = e^{-\frac{1}{5}t} \cdot e^C < C$$

$$|100 - B| = C e^{-\frac{1}{5}t}$$

$$100 - B = \pm C e^{-\frac{1}{5}t}$$

$$100 - B = C e^{-\frac{1}{5}t}$$

$$100 - C e^{-\frac{1}{5}t} = B$$

$$B = 100 - C e^{-\frac{1}{5}t}$$

$$20 = 100 - C e^{-\frac{1}{5}(0)}$$

$$-80 = -C$$

$$C = 80$$

$$B = 100 - 80 e^{-\frac{1}{5}t}$$

plug in
 $(0, 20)$
 (t, B)

Solve the below differential equation:

2) $y' - xy\cos(x^2) = 0$ given $y(0) = e$ a) Find general solution b) Find particular solution

$$y' - xy\cos(x^2) = 0$$

$$\frac{dy}{dx} - xy\cos(x^2) = 0$$

$$\frac{dy}{dx} = \frac{xy\cos(x^2)}{1}$$

$$\frac{dy}{y} = \frac{xy\cos(x^2)}{y} dx$$

$$\frac{dy}{y} = x\cos(x^2) dx$$

$$\int \frac{1}{y} dy = \int x\cos(x^2) dx$$

$$u = x^2$$

$$\frac{du}{dx} = 2x$$

$$2x dx = du$$

$$dx = \frac{du}{2x}$$

$$\int x \cos u \cdot \frac{du}{2x}$$

$$\int \frac{1}{y} dy = \frac{1}{2} \int \cos u du$$

$$\ln|y| = \frac{1}{2} \sin(x^2) + C$$

$$e^{\ln|y|} = e^{\frac{1}{2} \sin(x^2) + C} \quad \leftarrow a^m \cdot a^n = a^{m+n}$$

$$|y| = e^{\frac{1}{2} \sin(x^2)} \cdot [e^C] < C$$

$$y = \pm C e^{\frac{1}{2} \sin(x^2)} \quad \text{plug in } (0, e)$$

$$e = C e^{\frac{1}{2} \sin(0)^2}$$

$$e = C(1)$$

$$\underline{C = e}$$

$$y = (e)e^{\frac{1}{2} \sin(x^2)}$$

or

$$y = e^{1 + \frac{1}{2} \sin(x^2)}$$

$$\frac{1}{2} \int \cos u du$$

Solving Differential Equations: Additional Practice Problems

3) Solve the Differential Equation: $y \ln x^4 - xy' = 0$

$$y \ln x^4 - xy' = 0$$

$$y \ln x^4 - x \left(\frac{dy}{dx} \right) = 0$$

$$\frac{y \ln x^4}{1} = \frac{x dy}{dx}$$

$$x dy = y \ln x^4 dx$$

$$\int \frac{dy}{y} = \int \frac{\ln x^4}{x} dx$$

$$\int \frac{1}{y} dy = \int \frac{4 \ln x}{x} dx$$

$$u = \ln x \quad \begin{cases} dx = x du \\ \frac{du}{dx} = \frac{1}{x} \end{cases} \quad 4 \int u du$$

$$\int \frac{1}{y} dy = 4 \int u du$$

$$\ln |y| = 4 \cdot \frac{u^2}{2} + C$$

$$e^{\ln |y|} = e^{2(\ln x)^2 + C} \quad \leftarrow a^m \cdot a^n = a^{m+n}$$

$$|y| = e^{2(\ln x)^2} \cdot e^C \quad \boxed{C}$$

$$\boxed{y = \pm e^{2(\ln x)^2}}$$

$$y = Ce^{2(\ln x)^2}$$

u-sub

4) a) Find the general solution

$$yy' - 2e^{3x} = 0$$

$$\frac{ydy}{dx} - 2e^{3x} = 0$$

$$\frac{ydy}{dx} = \frac{2e^{3x}}{1}$$

$$ydy = 2e^{3x} dx$$

$$\int ydy = \int 2e^{3x} dx$$

$$\begin{aligned} u &= 3x & 3dx &= du \\ \frac{du}{dx} &= 3 & dx &= \frac{du}{3} \end{aligned}$$

$$2 \int e^u \cdot \frac{du}{3}$$

$$\int ydy = 2 \cdot \frac{1}{3} \int e^u du$$

$$\frac{y^2}{2} = \frac{2}{3} e^{3x} + C$$

b) Find the particular solution

$$yy' - 2e^{3x} = 0 \quad y(0) = 5$$

$$2\left(\frac{y^2}{2} = \frac{2}{3} e^{3x} + C\right)$$

$$y^2 = \frac{4}{3} e^{3x} + C$$

$$5^2 = \frac{4}{3} e^{3(0)} + C$$

$$25 = \frac{4}{3} + C$$

$$25 - \frac{4}{3} = C$$

$$\frac{71}{3} = C$$

$$y^2 = \frac{4}{3} e^{3x} + \frac{71}{3}$$

$$\sqrt{y^2} = \sqrt{\frac{4}{3} e^{3x} + \frac{71}{3}}$$

$$y = \sqrt{\frac{4}{3} e^{3x} + \frac{71}{3}}$$

or

~~$$y = -\sqrt{\frac{4}{3} e^{3x} + \frac{71}{3}}$$~~

Inverse Trig Integral Rules

a is a constant 1. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right) + C$ 2. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C$ 3. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec}\left(\frac{|u|}{a}\right) + C$

Completing the Square Steps:

1. Write in standard form: $x^2 + bx + c$
2. Add spaces $x^2 + bx + \underline{\quad} + c - \underline{\quad}$
3. Put $\left(\frac{b}{2}\right)^2$ into the spaces
4. Factor expression

$$\left(\frac{-6}{2}\right)^2 = 9 \quad \text{Ex. 4: } \int \frac{dx}{x^2 - 6x + 13}$$

$$x^2 - 6x + 13$$

$$\frac{x^2 - 6x + 9}{(x-3)^2 + 4} + 13 - 9$$

*Usually Arc Trig integral rules work best
when there are no variables in numerator

$$\begin{array}{l} \int \frac{dx}{(x-3)^2 + 2^2} \\ u = x-3 \\ \frac{du}{dx} = 1 \quad dx = du \\ a = 2 \end{array} \quad \boxed{\frac{1}{2} \arctan\left(\frac{x-3}{2}\right) + C}$$

u-sub arctrig Rule

$$\text{Ex. 5: } \int \frac{2x-5}{x^2+2x+2} dx = \int \frac{2x+2}{x^2+2x+2} dx + \int \frac{-7}{x^2+2x+2} dx$$

$u = x^2 + 2x + 2$ $u = x+1 \quad a = 1$
 $\frac{du}{dx} = 2x+2$ $\frac{du}{dx} = 1$
 $dx = \frac{du}{2x+2}$ $-7 \int \frac{1}{u^2+a^2} du$
 $\int \frac{1}{u} du$ $-7 \cdot \frac{1}{a} \arctan\left(\frac{u}{a}\right)$

$$\boxed{\ln|x^2+2x+2| - 7 \arctan\left(\frac{x+1}{1}\right) + C}$$

5.7 Notes : Integrals of Inverse Trig Functions

*Recall trig derivative Rules: $\frac{d}{dx} \arcsin u = \frac{u'}{\sqrt{1-u^2}}$ $\frac{d}{dx} \arctan u = \frac{u'}{1+u^2}$ $\frac{d}{dx} \operatorname{arcsec} u = \frac{u'}{|u|\sqrt{u^2-1}}$

Memorize the following formulas:

*($a = \text{constant}$)

$$1) \int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right) + C$$

$$2) \int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C$$

$$3) \int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec}\left(\frac{|u|}{a}\right) + C$$

$$\boxed{\text{Ex. 1}} \quad \int \frac{5}{x\sqrt{x^2-9}} dx$$

*this matches the arcsec rule exactly, so no need for u-substitution. Just plug in rule.

$$5 \int \frac{dx}{x\sqrt{x^2-(3)^2}} = 5 \cdot \frac{1}{3} \operatorname{arcsec}\left(\frac{|x|}{3}\right) + C$$

$$a=3 \rightarrow$$

$$= \boxed{\frac{5}{3} \operatorname{arcsec}\left(\frac{|x|}{3}\right) + C}$$

$$\boxed{\text{Ex. 2}} \quad \int \frac{1}{4+(x-1)^2} dx$$

*This matches arctan rule with no square root.

$$\int \frac{1}{(2)^2 + (x-1)^2} dx \quad \begin{aligned} a &= 2 \\ u &= x-1 \\ \frac{du}{dx} &= 1 \\ dx &= du \end{aligned}$$

$$\int \frac{1}{(a)^2 + (u)^2} du = \boxed{\frac{1}{2} \arctan\left(\frac{x-1}{2}\right) + C}$$

$$\boxed{\text{Ex. 3}} \quad \int \frac{1}{\sqrt{7-16x^2}} dx$$

*This form matches arcsin rule.

$$\int \frac{1}{\sqrt{(7)^2 - (4x)^2}} dx \quad \begin{aligned} a &= \sqrt{7} \\ u &= 4x \\ \frac{du}{dx} &= 4 \\ dx &= \frac{du}{4} \end{aligned}$$

$$\int \frac{1}{\sqrt{a^2 - u^2}} \cdot \frac{du}{4}$$

$$\frac{1}{4} \int \frac{du}{\sqrt{a^2 - u^2}} = \boxed{\frac{1}{4} \cdot \arcsin\left(\frac{4x}{\sqrt{7}}\right) + C}$$

5.7 Homework p.385 #1-41 odd (skip #7) 1/5

Inverse Trig Antiderivatives

$$11) \int \frac{t}{\sqrt{1-t^4}} dt \quad \left| \begin{array}{l} \int \frac{t}{\sqrt{(1)^2-(t^2)^2}} dt \\ a=1 \\ u=t^2 \\ \frac{du}{dt}=2t \end{array} \right| \quad \left| \begin{array}{l} \int \frac{t}{\sqrt{a^2-u^2}} \cdot \frac{du}{2t} = \frac{1}{2} \int \frac{du}{\sqrt{a^2-u^2}} \\ = \frac{1}{2} \cdot \arcsin\left(\frac{t^2}{1}\right) + C \\ = \boxed{\frac{1}{2} \arcsin(t^2) + C} \end{array} \right.$$

* matches arcsin rule

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C$$

$$13) \int \frac{e^{2x}}{4+e^{4x}} dx \quad \left| \begin{array}{l} \int \frac{e^{2x}}{(2)^2+(e^{2x})^2} dx \\ a=2 \\ u=e^{2x} \\ \frac{du}{dx}=e^{2x} \cdot 2 \end{array} \right| \quad \left| \begin{array}{l} \int \frac{e^{2x}}{a^2+u^2} \cdot \frac{du}{2e^{2x}} = \frac{1}{2} \int \frac{du}{a^2+u^2} \\ = \frac{1}{2} \cdot \frac{1}{2} \arctan\left(\frac{e^{2x}}{2}\right) + C \\ = \boxed{\frac{1}{4} \arctan\left(\frac{e^{2x}}{2}\right) + C} \end{array} \right.$$

* matches arctan rule

$$\int \frac{du}{a^2+u^2} = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C$$

$$15) \int \frac{1}{\sqrt{x} \sqrt{1-x}} dx \quad \left| \begin{array}{l} \int \frac{1}{\sqrt{x} \sqrt{(1)^2-(\sqrt{x})^2}} dx \\ a=1 \\ u=\sqrt{x} = x^{1/2} \\ \frac{du}{dx} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}} \\ dx = 2\sqrt{x} du \end{array} \right| \quad \left| \begin{array}{l} \int \frac{1}{\sqrt{k} \sqrt{a^2-u^2}} \cdot 2\sqrt{x} du = 2 \int \frac{du}{\sqrt{a^2-u^2}} \\ = 2 \cdot \arcsin\left(\frac{\sqrt{x}}{1}\right) + C \\ = \boxed{2 \arcsin(\sqrt{x}) + C} \end{array} \right.$$

* matches arcsin rule

b/c constant is positive
inside square root

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C$$

5.7 Homework (continued)

3/5

(1) 23) $\int_0^{\sqrt{3}/2} \frac{1}{1+4x^2} dx$

* This matches arctan rule

$$\begin{aligned} \int \frac{dx}{(1+(2x)^2)} & \quad a=1 \\ u=2x & \\ \frac{du}{dx}=2 & \end{aligned}$$

$$\begin{aligned} \left| \int \frac{1}{a^2+u^2} \cdot \frac{du}{2} \right. \\ \left. \frac{1}{2} \int \frac{du}{a^2+u^2} \right. \\ = \frac{1}{2} \cdot \frac{1}{2} \arctan\left(\frac{u}{1}\right) \left. \right|_0^{\sqrt{3}} \\ = \frac{1}{2} \arctan(\sqrt{3}) - \frac{1}{2} \arctan(0) \\ = \frac{1}{2}\left(\frac{\pi}{3}\right) - \frac{1}{2}(0) \\ = \boxed{\frac{\pi}{6}} \end{aligned}$$

(2) 25) $\int_0^{\sqrt{2}} \frac{\arcsinx}{\sqrt{1-x^2}} dx$

* Recall $\frac{d}{dx} \arcsin u = \frac{u'}{\sqrt{1-u^2}}$

$u=\arcsinx$

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \quad dx = \sqrt{1-x^2} du$$

$$\begin{aligned} \left| \int \frac{u}{\sqrt{1-x^2}} \cdot \sqrt{1-x^2} du \right. \\ \left. \int u du = \frac{u^2}{2} \right. \\ \frac{u^2}{2} \left. \right|_0^{\pi/4} = \frac{(\pi/4)^2}{2} - \frac{0}{2} \\ = \frac{1}{2}\left(\frac{\pi^2}{16}\right) = \boxed{\frac{\pi^2}{32}} \end{aligned}$$

(3) 29) $\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x} dx$

* This form matches arctan rule

$$\int \frac{\sin x}{(1+(\cos x)^2} dx$$

$a=1$

$u=\cos x$

$$\frac{du}{dx} = -\sin x \quad | \quad dx = \frac{du}{-\sin x}$$

$$\begin{aligned} \left| \int \frac{\sin x}{(a^2+u^2)} \cdot \frac{du}{-\sin x} \right. \\ \left. - \int \frac{du}{a^2+u^2} \right. \\ = -\frac{1}{2} \arctan\left(\frac{u}{1}\right) \left. \right|_0^{-1} \\ = -\arctan(-1) - (-\arctan 0) \\ = -\left(-\frac{\pi}{4} + 0\right) \end{aligned}$$

$$= \boxed{\frac{\pi}{4}}$$

5.7 Homework (continued)

4/5

$$31) \int_0^2 \frac{dx}{x^2 - 2x + 2}$$

* complete the square
in denominator

$$x^2 - 2x + \underline{\quad} + 2 = \underline{\quad}$$

$$\left(\frac{b}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = 1$$

$$x^2 - 2x + \underline{1} + 2 = \underline{1}$$

$$(x-1)^2 + 1$$

$$\int \frac{dx}{(x-1)^2 + 1^2}$$

$$u = x-1 \quad du = dx \\ \frac{du}{dx} = 1$$

$$\int \frac{du}{u^2 + a^2} du$$

$$\text{if } x=0, u=0-1=-1$$

$$\text{if } x=2, u=2-1=1$$

$$= \frac{1}{1} \arctan\left(\frac{u}{1}\right) \Big|_{-1}^1$$

$$= \arctan(1) - \arctan(-1)$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{2\pi}{4} = \boxed{\frac{\pi}{2}}$$

$$33) \int \frac{2x}{x^2 + 6x + 13} dx$$

$$x^2 + 6x + \underline{\quad} + 13 = \underline{\quad}$$

$$\left(\frac{b}{2}\right)^2 = \left(\frac{6}{2}\right)^2 = 9$$

$$x^2 + 6x + \underline{9} + 13 = \underline{9}$$

$$(x+3)^2 + 4$$

$$\int \frac{2x}{(x+3)^2 + 4} dx$$

* arctan rule alone
will not be sufficient.
Try u-substitution
and split up equation

$$u = (x+3)^2 + 4$$

$$\frac{du}{dx} = 2(x+3)(1) + 0$$

$$dx = \frac{du}{2x+6}$$

$$\int \frac{2x}{(x+3)^2 + 4} dx = \int \frac{2x+6}{(x+3)^2 + 4} dx - \int \frac{6}{(x+3)^2 + 4} dx$$

(u-substitution)

(arctan rule)

$$\int \frac{2x+6}{u} \cdot \frac{du}{2x+6}$$

$$\int \frac{1}{u} du$$

$$\ln|u| + C$$

$$= \ln|(x+3)^2 + 4| + C$$

$$-6 \int \frac{dx}{(x+3)^2 + (2)^2}$$

$$u = x+3 \quad a = 2$$

$$\frac{du}{dx} = 1$$

$$-6 \int \frac{du}{u^2 + a^2} = -6 \cdot$$

$$= -6 \cdot \frac{1}{2} \arctan\left(\frac{x+3}{2}\right) + C$$

$$= \boxed{\ln|x^2 + 6x + 13| - 3 \arctan\left(\frac{x+3}{2}\right) + C}$$

$$35) \int \frac{1}{\sqrt{-x^2 - 4x}} dx$$

$$-(x^2 + 4x + \underline{\quad}) + \underline{\quad}$$

$$\left(\frac{4}{2}\right)^2 = 2^2 = 4$$

$$-(x^2 + 4x + \underline{4}) + \underline{4}$$

$$\int \frac{1}{\sqrt{-(x+2)^2 + 4}} dx$$

$$\int \frac{dx}{\sqrt{4 - (x+2)^2}}$$

* This form matches
arc sin rule

$$\int \frac{dx}{\sqrt{(2)^2 - (x+2)^2}}$$

$$a = 2$$

$$u = x+2$$

$$\frac{du}{dx} = 1$$

$$dx = du$$

$$\int \frac{du}{\sqrt{a^2 - u^2}}$$

$$= \boxed{\arcsin\left(\frac{x+2}{2}\right) + C}$$

5.7 Homework (continued)

5/5

$$37) \int \frac{x+2}{\sqrt{-x^2-4x}} dx$$

$$u = -x^2 - 4x$$

$$\frac{du}{dx} = -2x - 4$$

$$dx = \frac{du}{-2x-4}$$

$$dx = \frac{du}{-2(x+2)}$$

$$\begin{aligned} & \left| \int \frac{x+2}{\sqrt{u}} \cdot \frac{du}{-2(x+2)} \right| \\ &= \frac{1}{2} \int \frac{1}{\sqrt{u}} du \\ &= \frac{1}{2} \int u^{-1/2} du \end{aligned}$$

$$-\frac{1}{2} \left(\frac{u^{1/2}}{1/2} \right) + C$$

$$-\frac{1}{2}(2) u^{1/2} + C$$

$$= \sqrt{-x^2-4x} + C$$

$$39) \int_2^3 \frac{2x-3}{\sqrt{4x-x^2}} dx$$

$$u = 4x - x^2$$

$$\frac{du}{dx} = 4 - 2x = -2x + 4$$

$$dx = \frac{du}{-(2x-4)}$$

* Split up equation
to allow u-substitution
to work

$$\begin{aligned} & \int \frac{dx-3}{\sqrt{4x-x^2}} dx = \int \frac{2x-4}{\sqrt{4x-x^2}} dx + \int \frac{1}{\sqrt{4x-x^2}} dx \\ & \quad \text{(a-substitution)} \quad \text{(arcsin rule)} \\ & \left| \int \frac{2x-4}{\sqrt{u}} \cdot \frac{du}{-(2x-4)} \right| - (x^2 - 4x + 4) + \frac{4}{(2)^2} = 2^2 = 4 \\ & \quad - \int u^{-1/2} du \\ & \quad = -\frac{u^{1/2}}{1/2} \\ & \quad = -2 \sqrt{4x-x^2} + \arcsin\left(\frac{x-2}{2}\right) \quad \left| \begin{array}{l} a=2 \quad u=x-2 \quad \frac{du}{dx}=1 \quad dx=du \\ \int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) \end{array} \right. \\ & \quad \left. \begin{aligned} & \left[-2 \sqrt{4x-x^2} + \arcsin\left(\frac{x-2}{2}\right) \right]_2^3 = -2\sqrt{3} + \frac{\pi}{6} - (-4 + 0) \\ & \quad = -2\sqrt{3} + \frac{\pi}{6} + 4 \approx 1.059 \end{aligned} \right. \end{aligned}$$

$$41) \int \frac{x}{x^4+2x^2+2} dx$$

$$x^4+2x^2+ \underline{\quad} + 2 - \underline{\quad}$$

$$\left(\frac{2}{2}\right)^2 = 1^2 = 1$$

$$x^4+2x^2+\underline{1}+2-\underline{1}$$

$$(x^2+1)^2 + 1$$

$$\int \frac{x}{(x^2+1)^2+(1)^2} dx$$

* This form matches
arctan rule

$$u = x^2 + 1 \quad a = 1$$

$$\frac{du}{dx} = 2x$$

$$dx = \frac{du}{2x}$$

$$\int \frac{x}{u^2+a^2} \cdot \frac{du}{2x}$$

$$\frac{1}{2} \int \frac{du}{u^2+a^2}$$

$$\frac{1}{2} \cdot \frac{1}{a} \arctan\left(\frac{u^2+1}{1}\right) + C$$

$$= \boxed{\frac{1}{2} \arctan(x^2+1) + C}$$



Calculus Ch. 5.7 Notes

Integrals of Inverse Trig Functions

Key

 $a = \text{constant}$ $u = \text{Variable expression}$

Recall Rules for Inverse Trig Derivatives:

$$\frac{d}{dx} \arcsin u = \frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx} \arccos u = -\frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx} \arctan u = \frac{u'}{1+u^2}$$

$$\frac{d}{dx} \operatorname{arc cot} u = -\frac{u'}{1+u^2}$$

$$\frac{d}{dx} \operatorname{arc sec} u = \frac{u'}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx} \operatorname{arc csc} u = -\frac{u'}{|u|\sqrt{u^2-1}}$$

Inverse Trig Integral Rules:

a is a constant

$$1. \int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C$$

$$2. \int \frac{du}{a^2+u^2} = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C$$

$$3. \int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a} \operatorname{arcsec}\left(\frac{|u|}{a}\right) + C$$

$$\text{Ex. 1: } \int \frac{5}{x\sqrt{x^2-9}} dx$$

$$a=3$$

$$u=x$$

$$\frac{du}{dx} = 1$$

$$dx=du$$

$$5 \int \frac{1}{u\sqrt{u^2-9}} du \rightarrow 5 \cdot \frac{1}{3} \operatorname{arcsec}\left(\frac{|x|}{3}\right) + C$$

$$\boxed{\frac{5}{3} \operatorname{arcsec}\left(\frac{|x|}{3}\right) + C}$$

$$\text{Ex. 2: } \int \frac{1}{4+(x-1)^2} dx$$

$$a=2$$

$$u=x-1$$

$$\int \frac{1}{a^2+u^2} du = \frac{1}{2} \arctan\left(\frac{x-1}{2}\right) + C$$

$$\int \frac{1}{(4)^2+(x-1)^2} dx$$

$$\frac{du}{dx} = 1$$

$$dx=du$$

$$\text{Ex. 3: } \int \frac{1}{\sqrt{7-16x^2}} dx$$

$$a=\sqrt{7}$$

$$u=4x$$

$$\frac{du}{dx} = 4$$

$$4dx=du$$

$$dx=\frac{du}{4}$$

$$\int \frac{1}{\sqrt{a^2-u^2}} du \rightarrow \frac{1}{4} \int \frac{1}{\sqrt{a^2-u^2}} du$$

$$\boxed{\frac{1}{4} \cdot \arcsin\left(\frac{4x}{\sqrt{7}}\right) + C}$$

Inverse Trig Integral Rules

a is a constant 1. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right) + C$ 2. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C$ 3. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec}\left(\frac{|u|}{a}\right) + C$

Completing the Square Steps:

1. Write in standard form: $x^2 + bx + c$
2. Add spaces $x^2 + bx + \underline{\quad} + c - \underline{\quad}$
3. Put $\left(\frac{b}{2}\right)^2$ into the spaces
4. Factor expression

Ex. 4: $\int \frac{dx}{x^2 - 6x + 13}$ $\left(\frac{b}{2}\right)^2 \rightarrow \left(\frac{-6}{2}\right)^2 = 9$

$$\begin{aligned} & x^2 - 6x + 13 \\ & x^2 - 6x + \underline{9} + 13 - \underline{9} \\ & (x-3)(x-3) \\ & (x-3)^2 + 4 \end{aligned}$$

$$\begin{aligned} & \int \frac{dx}{(x-3)^2 + 4} \rightarrow \int \frac{dx}{(x-3)^2 + (2)^2} \\ & a=2 \\ & u=x-3 \\ & \frac{du}{dx}=1 \\ & dx=du \\ & \int \frac{du}{u^2 + a^2} = \frac{1}{2} \arctan\left(\frac{x-3}{2}\right) + C \\ & \boxed{\frac{1}{2} \arctan\left(\frac{x-3}{2}\right) + C} \end{aligned}$$

Ex. 5: $\int \frac{2x-5}{x^2+2x+2} dx$

41) $\int \frac{x}{x^4 + 2x^2 + 2} dx$

$$\begin{aligned} & \int \frac{x}{(x^2+1)^2 + 1} dx \rightarrow \int \frac{x}{(x^2+1)^2 + (1)^2} dx \\ & \int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C \end{aligned}$$

complete the square: $\left(\frac{b}{2}\right)^2 \rightarrow \left(\frac{2}{2}\right)^2 = 1$

$$\begin{aligned} & x^4 + 2x^2 + 2 \\ & x^4 + 2x^2 + \underline{1} + 2 - \underline{1} \\ & (x^2+1)(x^2+1) \\ & (x^2+1)^2 + 1 \end{aligned}$$

$$\begin{aligned} & a=1 \\ & u=x^2+1 \\ & \frac{du}{dx}=2x \\ & 2x dx = du \\ & dx = \frac{du}{2x} \\ & \int \frac{x}{u^2 + a^2} \cdot \frac{du}{2x} \\ & \frac{1}{2} \int \frac{du}{u^2 + a^2} = \\ & \frac{1}{2} \cdot \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C \\ & \boxed{\frac{1}{2} \arctan(x^2+1) + C} \end{aligned}$$

5.7 Inverse Trig Antiderivatives p. 380 #5-41 odd

9) $\int \frac{t}{t^4+25} dt$

$$* \int \frac{du}{a^2+u^2} = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C$$

$$\begin{aligned} \int \frac{t}{(5)^2+(t^2)^2} dt & \quad u=t^2 \quad \left| dt = \frac{du}{2t} \right| \quad \int \frac{t}{a^2+u^2} \cdot \frac{du}{2t} = \frac{1}{2} \int \frac{du}{a^2+u^2} = \\ & \quad \frac{du}{dt} = 2t \quad \left| = \frac{1}{2} \cdot \frac{1}{5} \arctan\left(\frac{t^2}{5}\right) + C \right. \\ & \quad \boxed{\left. = \frac{1}{10} \arctan\left(\frac{t^2}{5}\right) + C \right]} \end{aligned}$$

13) $\int \frac{\sec^2 x}{\sqrt{25-\tan^2 x}} dx$

$$* \int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C$$

$$\begin{aligned} \int \frac{\sec^2 x}{\sqrt{5^2-(\tan x)^2}} dx & \quad a=5 \\ & \quad u=\tan x \quad \left| dx = \frac{du}{\sec^2 x} \right. \\ & \quad \frac{du}{dx} = \sec^2 x \quad \left| \int \frac{\sec^2 x}{\sqrt{a^2-u^2}} \cdot \frac{du}{\sec^2 x} \right. \\ & \quad \boxed{\left. = \arcsin\left(\frac{\tan x}{5}\right) + C \right]} \end{aligned}$$

15) $\int \frac{1}{\sqrt{x} \sqrt{1-x}} dx$

$$* \int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C$$

$$\begin{aligned} \int \frac{1}{\sqrt{x} \cdot \sqrt{(1)^2-(\sqrt{x})^2}} dx & \quad a=1 \\ & \quad u=\sqrt{x} \quad \left| dx = 2\sqrt{x} du \right. \\ & \quad \frac{du}{dx} = \frac{1}{2}x^{-1/2} \quad \left| \int \frac{1}{\sqrt{x} \sqrt{a^2-u^2}} \cdot 2\sqrt{x} du \right. \\ & \quad \frac{du}{dx} = \frac{1}{2\sqrt{x}} \quad \left| 2 \int \frac{du}{\sqrt{a^2-u^2}} \right. \\ & \quad \boxed{\left. 2 \arcsin\left(\frac{\sqrt{x}}{1}\right) + C \right.} \\ & \quad \boxed{2 \arcsin\sqrt{x} + C} \end{aligned}$$

[5.7]

$$17) \int \frac{x-3}{x^2+1} dx = \int \frac{x}{x^2+1} dx - \int \frac{3}{x^2+1} dx$$

(u-sub) (arctan)

$\begin{aligned} u &= x^2 + 1 & \int \frac{x}{u} \cdot \frac{du}{2x} & \quad \begin{aligned} a &= 1 \\ u &= x & du &= dx \\ \frac{du}{dx} &= 2x & \frac{du}{dx} &= 1 \end{aligned} \\ \frac{du}{dx} &= 2x & \int \frac{1}{u} du & \quad \int \frac{3}{(x^2+1)^2} dx \\ dx &= \frac{du}{2x} & = \frac{1}{2} \int \frac{1}{u} du & \quad \int \frac{3}{u^2+a^2} du = 3 \cdot \frac{1}{2} \arctan\left(\frac{x}{a}\right) + C \\ & & = \frac{1}{2} \ln|u| & \quad = \boxed{\frac{1}{2} \ln|x^2+1| - 3 \arctan x + C} \end{aligned}$

$$19) \int \frac{x+5}{\sqrt{9-(x-3)^2}} dx = \int \frac{x-3}{\sqrt{9-(x-3)^2}} dx + \int \frac{8}{\sqrt{9-(x-3)^2}} dx$$

$\begin{aligned} u &= 9-(x-3)^2 & \int \frac{x-3}{u^{1/2}} \cdot \frac{du}{-2(x-3)} & \quad \begin{aligned} \int \frac{8}{\sqrt{(3)^2-(x-3)^2}} du &= \int \frac{8}{\sqrt{a^2-u^2}} du \\ \frac{du}{dx} &= 0 - 2(x-3) = -2x+6 & u &= 3 \\ \frac{du}{dx} &= -2(x-3) & \frac{du}{dx} &= 1 \end{aligned} \\ & & -\frac{1}{2} \int \frac{du}{u^{1/2}} &= -\frac{1}{2} \int u^{-1/2} du & \quad \begin{aligned} a &= 3 \\ u &= x-3 \\ \frac{du}{dx} &= 1 \end{aligned} \\ & & -\frac{1}{2} \frac{u^{1/2}}{1/2} &+ 8 \cdot \arcsin\left(\frac{x-3}{3}\right) + C & \end{aligned}$

$$31) \int_0^{\pi/2} \frac{\arcsin x}{\sqrt{1-x^2}} dx = \boxed{-\sqrt{9-(x-3)^2} + 8 \arcsin\left(\frac{x-3}{3}\right) + C}$$

convert bounds:

$x=0, u=\arcsin 0=0$
 $x=\pi/2, u=\arcsin \pi/2=\pi/4$

$\begin{aligned} u &= \arcsin x & \int \frac{u}{\sqrt{1-x^2}} \sqrt{1-x^2} du \\ \frac{du}{dx} &= \frac{1}{\sqrt{1-x^2}} & = \frac{u^2}{2} \Big|_{0}^{\pi/4} = \frac{(\pi/4)^2}{2} - 0 = \frac{\pi^2}{16} = \boxed{\frac{\pi^2}{32} \approx 0.308} \\ dx &= \sqrt{1-x^2} du & & \end{aligned}$

Complete the square

$$33) \int_0^2 \frac{dx}{x^2 - 2x + 2}$$

$$\int_0^2 \frac{dx}{x^2 - 2x + 1 + 2 - 1}$$

$$\left| \int \frac{dx}{(x-1)^2 + 1^2} \right|$$

$u = x-1 \quad a = 1$
 $\frac{du}{dx} = 1 \quad dx = du$

$$\left[\frac{1}{2} \arctan(x-1) \right]_0^2$$

$$\arctan(1) - \arctan(-1)$$

principal value
 \arctan
 $-\pi/2 < y < \pi/2$

$$\frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{2\pi}{4} = \boxed{\frac{\pi}{2}}$$

$$35) \int \frac{2x}{x^2 + 6x + 13} dx$$

$$\int \frac{-6}{x^2 + 6x + 13 - 9} dx$$

$$\left| \begin{array}{l} \int \frac{-6}{x^2 + 6x + 9 + 13 - 9} dx \\ \int \frac{-6}{(x+3)^2 + 4} dx \end{array} \right|$$

$u = x+3 \quad a = 2$
 $\frac{du}{dx} = 1$

$$\int \frac{-6}{(x+3)^2 + 2^2} dx$$

$$\left(-6 \int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C \right)$$

$$= -6 \cdot \frac{1}{2} \arctan\left(\frac{x+3}{2}\right)$$

$$35) \int \frac{2x}{x^2 + 6x + 13} dx = \int \frac{2x+6}{x^2 + 6x + 13} dx + \int \frac{-6}{x^2 + 6x + 13}$$

$$u = x^2 + 6x + 13$$

$$\frac{du}{dx} = 2x+6$$

$$dx = \frac{du}{2x+6}$$

$$\int \frac{2x+6}{u} \frac{du}{2x+6}$$

$$\ln|u| + -3 \arctan\left(\frac{x+3}{2}\right)$$

$$\boxed{\ln|x^2 + 6x + 13| - 3 \arctan\left(\frac{x+3}{2}\right) + C}$$

$$37) \int \frac{1}{\sqrt{-x^2 - 4x}} dx$$

$$\int \frac{1}{\sqrt{-(x^2 + 4x + 4) + 4}} dx$$

$$\int \frac{1}{\sqrt{-(x+2)^2 + 2^2}} dx$$

$$\int \frac{1}{\sqrt{4 - (x+2)^2}} dx$$

$$a = 2 \quad u = x+2$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right) + C$$

$$39) \int_2^3 \frac{2x-3}{\sqrt{4x-x^2}} dx = \int \frac{2x-4}{\sqrt{4x-x^2}} dx + \int \frac{1}{\sqrt{4x-x^2}} dx$$

$\sqrt{-(x^2-4x+4)+4}$

$u = 4x - x^2$

$\frac{du}{dx} = -2x + 4$

$dx = \frac{du}{-2x+4}$

$\int \frac{2x-4}{u^{1/2} - (2x-4)} du$

$= \int u^{-1/2} du$

$= \frac{u^{1/2}}{\frac{1}{2}}$

$= \left[-2(4x-x^2)^{1/2} + \arcsin\left(\frac{x-2}{2}\right) \right]_2^3 = \boxed{4-2\sqrt{3} + \frac{\pi}{6}}$

$= -2(3)^{1/2} + \arcsin(\frac{1}{2}) - \left[-2(4)^{1/2} + \arcsin(0) \right] = -2\sqrt{3} + \frac{\pi}{6} + 4 \approx 0$

$$41) \int \frac{x}{x^4+2x^2+2} dx = \int \frac{x}{x^4+2x^2+1+2-1} dx$$

$\int \frac{x}{(x^2+1)^2+(1)^2} dx \quad u = x^2+1 \quad du = 2x dx \quad \frac{du}{dx} = 2x \quad a = 1$

$\int \frac{x}{u^2+a^2} \cdot \frac{du}{2x} = \frac{1}{2} \int \frac{du}{a^2+u^2}$

$= \frac{1}{2} \cdot \frac{1}{1} \arctan\left(\frac{x^2+1}{1}\right) + C$

$= \boxed{\frac{1}{2} \arctan(x^2+1) + C}$