6.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding a General Solution Using Separation of Variables In Exercises 1–14, find the general solution of the differential equation.

1. $\dfrac{dy}{dx} = \dfrac{x}{y}$

2. $\dfrac{dy}{dx} = \dfrac{3x^2}{y^2}$

3. $x^2 + 5y\dfrac{dy}{dx} = 0$

4. $\dfrac{dy}{dx} = \dfrac{6 - x^2}{2y^3}$

5. $\dfrac{dr}{ds} = 0.75r$

6. $\dfrac{dr}{ds} = 0.75s$

7. $(2 + x)y' = 3y$

8. $xy' = y$

9. $yy' = 4 \sin x$

10. $yy' = -8 \cos(\pi x)$

11. $\sqrt{1 - 4x^2}\, y' = x$

12. $\sqrt{x^2 - 16}\, y' = 11x$

13. $y \ln x - xy' = 0$

14. $12yy' - 7e^x = 0$

Finding a Particular Solution Using Separation of Variables In Exercises 15–24, find the particular solution that satisfies the initial condition.

Differential Equation	Initial Condition
15. $yy' - 2e^x = 0$	$y(0) = 3$
16. $\sqrt{x} + \sqrt{y}\, y' = 0$	$y(1) = 9$
17. $y(x + 1) + y' = 0$	$y(-2) = 1$
18. $2xy' - \ln x^2 = 0$	$y(1) = 2$
19. $y(1 + x^2)y' - x(1 + y^2) = 0$	$y(0) = \sqrt{3}$
20. $y\sqrt{1 - x^2}\, y' - x\sqrt{1 - y^2} = 0$	$y(0) = 1$
21. $\dfrac{du}{dv} = uv \sin v^2$	$u(0) = 1$
22. $\dfrac{dr}{ds} = e^{r - 2s}$	$r(0) = 0$
23. $dP - kP\, dt = 0$	$P(0) = P_0$
24. $dT + k(T - 70)\, dt = 0$	$T(0) = 140$

Finding a Particular Solution In Exercises 25–28, find an equation of the graph that passes through the point and has the given slope.

25. $(0, 2),\quad y' = \dfrac{x}{4y}$

26. $(1, 1),\quad y' = -\dfrac{9x}{16y}$

27. $(9, 1),\quad y' = \dfrac{y}{2x}$

28. $(8, 2),\quad y' = \dfrac{2y}{3x}$

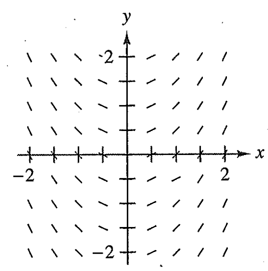
Using Slope In Exercises 29 and 30, find all functions f having the indicated property.

29. The tangent to the graph of f at the point (x, y) intersects the x-axis at $(x + 2, 0)$.

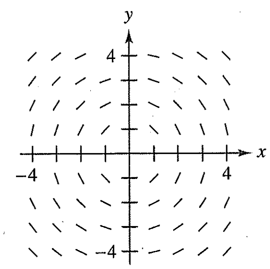
30. All tangents to the graph of f pass through the origin.

Slope Field In Exercises 31 and 32, sketch a few solutions of the differential equation on the slope field and then find the general solution analytically. To print an enlarged copy of the graph, go to *MathGraphs.com*.

31. $\dfrac{dy}{dx} = x$

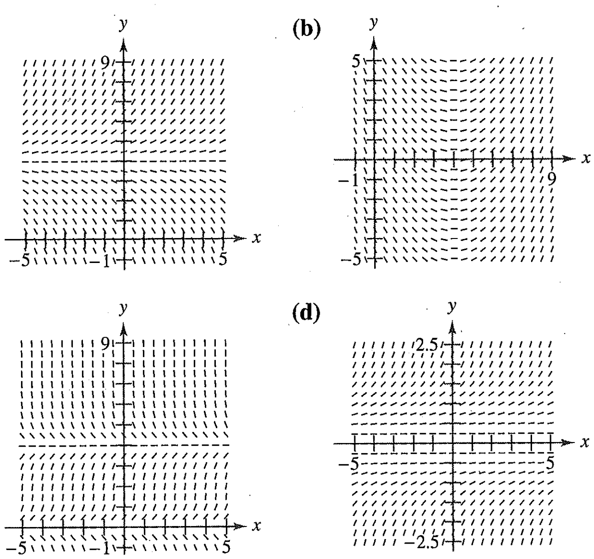


32. $\dfrac{dy}{dx} = -\dfrac{x}{y}$



Slope Field In Exercises 33–36, (a) write a differential equation for the statement, (b) match the differential equation with a possible slope field, and (c) verify your result by using a graphing utility to graph a slope field for the differential equation. [The slope fields are labeled (a), (b), (c), and (d).] To print an enlarged copy of the graph, go to *MathGraphs.com*.

(a) (b) (c) (d)



33. The rate of change of y with respect to x is proportional to the difference between y and 4.

34. The rate of change of y with respect to x is proportional to the difference between x and 4.

35. The rate of change of y with respect to x is proportional to the product of y and the difference between y and 4.

36. The rate of change of y with respect to x is proportional to y^2.

37. **Radioactive Decay** The rate of decomposition of radioactive radium is proportional to the amount present at any time. The half-life of radioactive radium is 1599 years. What percent of a present amount will remain after 50 years?

38. Chemical Reaction In a chemical reaction, a certain compound changes into another compound at a rate proportional to the unchanged amount. There is 40 grams of the original compound initially and 35 grams after 1 hour. When will 75 percent of the compound be changed?

39. Weight Gain A calf that weighs 60 pounds at birth gains weight at the rate

$$\frac{dw}{dt} = k(1200 - w)$$

where w is weight in pounds and t is time in years.

(a) Solve the differential equation.

(b) Use a graphing utility to graph the particular solutions for $k = 0.8, 0.9$, and 1.

(c) The animal is sold when its weight reaches 800 pounds. Find the time of sale for each of the models in part (b).

(d) What is the maximum weight of the animal for each of the models in part (b)?

40. Weight Gain A calf that weighs w_0 pounds at birth gains weight at the rate $dw/dt = 1200 - w$, where w is weight in pounds and t is time in years. Solve the differential equation.

Finding Orthogonal Trajectories **In Exercises 41–46, find the orthogonal trajectories of the family. Use a graphing utility to graph several members of each family.**

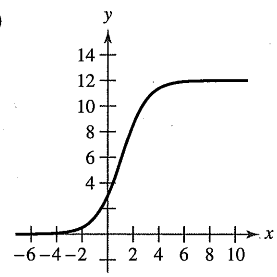
41. $x^2 + y^2 = C$ **42.** $x^2 - 2y^2 = C$

43. $x^2 = Cy$ **44.** $y^2 = 2Cx$

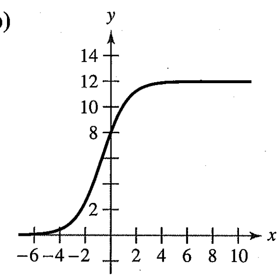
45. $y^2 = Cx^3$ **46.** $y = Ce^x$

Matching **In Exercises 47–50, match the logistic equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

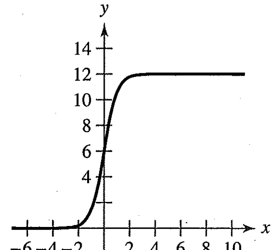
(a)



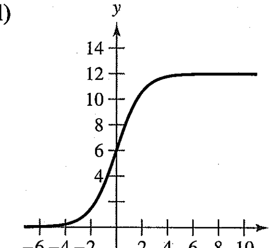
(b)



(c)



(d)



47. $y = \dfrac{12}{1 + e^{-x}}$ **48.** $y = \dfrac{12}{1 + 3e^{-x}}$

49. $y = \dfrac{12}{1 + \frac{1}{2}e^{-x}}$ **50.** $y = \dfrac{12}{1 + e^{-2x}}$

Using a Logistic Equation **In Exercises 51 and 52, the logistic equation models the growth of a population. Use the equation to (a) find the value of k, (b) find the carrying capacity, (c) find the initial population, (d) determine when the population will reach 50% of its carrying capacity, and (e) write a logistic differential equation that has the solution $P(t)$.**

51. $P(t) = \dfrac{2100}{1 + 29e^{-0.75t}}$ **52.** $P(t) = \dfrac{5000}{1 + 39e^{-0.2t}}$

Using a Logistic Differential Equation **In Exercises 53 and 54, the logistic differential equation models the growth rate of a population. Use the equation to (a) find the value of k, (b) find the carrying capacity, (c) graph a slope field using a computer algebra system, and (d) determine the value of P at which the population growth rate is the greatest.**

53. $\dfrac{dP}{dt} = 3P\left(1 - \dfrac{P}{100}\right)$ **54.** $\dfrac{dP}{dt} = 0.1P - 0.0004P^2$

Solving a Logistic Differential Equation **In Exercises 55–58, find the logistic equation that passes through the given point.**

55. $\dfrac{dy}{dt} = y\left(1 - \dfrac{y}{36}\right)$, $(0, 4)$ **56.** $\dfrac{dy}{dt} = 2.8y\left(1 - \dfrac{y}{10}\right)$, $(0, 7)$

57. $\dfrac{dy}{dt} = \dfrac{4y}{5} - \dfrac{y^2}{150}$, $(0, 8)$ **58.** $\dfrac{dy}{dt} = \dfrac{3y}{20} - \dfrac{y^2}{1600}$, $(0, 15)$

59. Endangered Species A conservation organization releases 25 Florida panthers into a game preserve. After 2 years, there are 39 panthers in the preserve. The Florida preserve has a carrying capacity of 200 panthers.

(a) Write a logistic equation that models the population of panthers in the preserve.

(b) Find the population after 5 years.

(c) When will the population reach 100?

(d) Write a logistic differential equation that models the growth rate of the panther population. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answer.

(e) At what time is the panther population growing most rapidly? Explain.

60. Bacteria Growth At time $t = 0$, a bacterial culture weighs 1 gram. Two hours later, the culture weighs 4 grams. The maximum weight of the culture is 20 grams.

(a) Write a logistic equation that models the weight of the bacterial culture.

(b) Find the culture's weight after 5 hours.

(c) When will the culture's weight reach 18 grams?

(d) Write a logistic differential equation that models the growth rate of the culture's weight. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answer.

(e) At what time is the culture's weight increasing most rapidly? Explain.

WRITING ABOUT CONCEPTS

61. **Separation of Variables** In your own words, describe how to recognize and solve differential equations that can be solved by separation of variables.

62. **Mutually Orthogonal** In your own words, describe the relationship between two families of curves that are mutually orthogonal.

63. **Finding a Derivative** Show that if

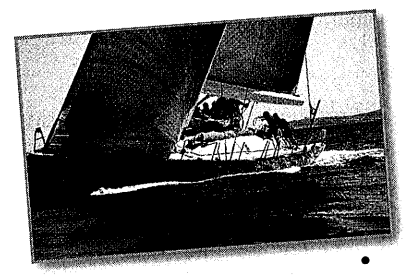
$$y = \frac{1}{1 + be^{-kt}}$$

then

$$\frac{dy}{dt} = ky(1 - y).$$

64. **Point of Inflection** For any logistic growth curve, show that the point of inflection occurs at $y = L/2$ when the solution starts below the carrying capacity L.

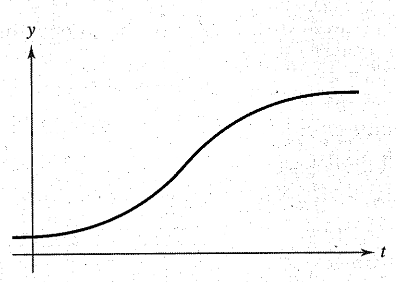
65. **Sailing** Ignoring resistance, a sailboat starting from rest accelerates (dv/dt) at a rate proportional to the difference between the velocities of the wind and the boat.



(a) The wind is blowing at 20 knots, and after 1 half-hour, the boat is moving at 10 knots. Write the velocity v as a function of time t.

(b) Use the result of part (a) to write the distance traveled by the boat as a function of time.

66. **HOW DO YOU SEE IT?** The growth of a population is modeled by a logistic equation as shown in the graph below. What happens to the rate of growth as the population increases? What do you think causes this to occur in real-life situations, such as animal or human populations?



Determining if a Function Is Homogeneous **In Exercises 67–74, determine whether the function is homogeneous, and if it is, determine its degree. A function $f(x, y)$ is *homogeneous of degree n* if $f(tx, ty) = t^n f(x, y)$.**

67. $f(x, y) = x^3 - 4xy^2 + y^3$

68. $f(x, y) = x^3 + 3x^2y^2 - 2y^2$

69. $f(x, y) = \dfrac{x^2y^2}{\sqrt{x^2 + y^2}}$

70. $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$

71. $f(x, y) = 2 \ln xy$

72. $f(x, y) = \tan(x + y)$

73. $f(x, y) = 2 \ln \dfrac{x}{y}$

74. $f(x, y) = \tan \dfrac{y}{x}$

Solving a Homogeneous Differential Equation **In Exercises 75–80, solve the homogeneous differential equation in terms of x and y. A *homogeneous differential equation* is an equation of the form $M(x, y)\,dx + N(x, y)\,dy = 0$, where M and N are homogeneous functions of the same degree. To solve an equation of this form by the method of separation of variables, use the substitutions $y = vx$ and $dy = x\,dv + v\,dx$.**

75. $(x + y)\,dx - 2x\,dy = 0$

76. $(x^3 + y^3)\,dx - xy^2\,dy = 0$

77. $(x - y)\,dx - (x + y)\,dy = 0$

78. $(x^2 + y^2)\,dx - 2xy\,dy = 0$

79. $xy\,dx + (y^2 - x^2)\,dy = 0$

80. $(2x + 3y)\,dx - x\,dy = 0$

True or False? **In Exercises 81–83, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

81. The function $y = 0$ is always a solution of a differential equation that can be solved by separation of variables.

82. The differential equation $y' = xy - 2y + x - 2$ can be written in separated variables form.

83. The families $x^2 + y^2 = 2Cy$ and $x^2 + y^2 = 2Kx$ are mutually orthogonal.

PUTNAM EXAM CHALLENGE

84. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).

This problem was composed by the Committee on the Putnam Prize Competition.