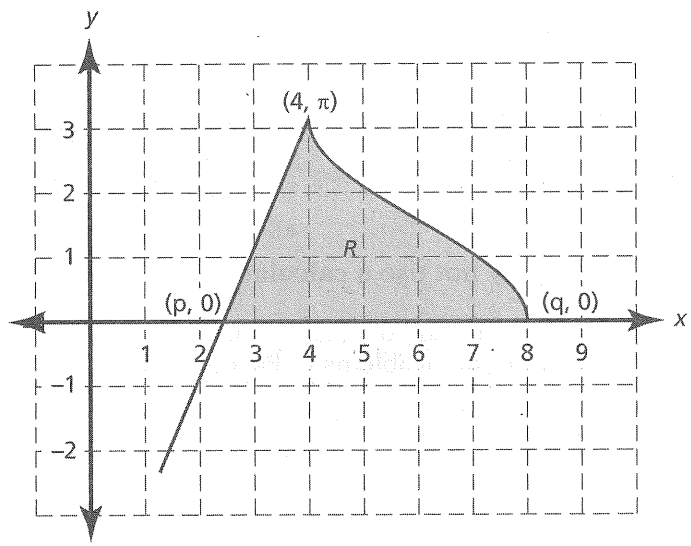
# Section II
## Free-Response Questions
## Time: 1 hour and 30 minutes
## Number of Problems: 6

## Part A
## Time: 45 minutes
## Number of Problems: 3

**You may use a calculator for any problem in this section.**

1. Region $R$ is bounded by the functions $f(x) = 2(x-4)+\pi$, $g(x) = \cos^{-1}\left(\frac{x}{2}-3\right)$, and the $x$-axis as shown in the figure to the right.
   a. What is the area of region $R$?
   b. Find the volume of the solid generated when region $R$ is rotated about the $x$-axis.
   c. Find all values $c$ for $f(x)$ and $g(x)$ in the closed interval $p \le c \le q$ for which each function equals the average value in the indicated interval.




2. A particle is moving along the $x$-axis with velocity $v(t) = \ln(t+3) - e^{\frac{t}{2}-1}(\cos t)$, for $0 \le t \le 8$. The initial position of the particle is $-1.6$.
   a. At what time(s) in the open interval $0 < t < 8$ does the particle change direction? Justify your answer.
   b. Where is the particle when it is farthest to the left?
   c. How far does the particle travel in the interval $3 \le t \le 6$?
   d. At what times in the closed interval, $0 \le t \le 8$, is the speed of the particle decreasing? Justify your answer.

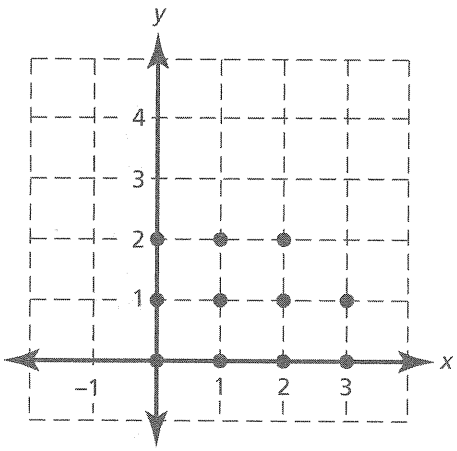
3. A large flea market is held at the local fairgrounds on the first Saturday of each month. The rates at which people enter and leave the fairgrounds are recorded for a 3-hour period beginning when the market is open to the public. The rate at which people arrive is modeled by the function $A(t) = 45 \sin(0.03t - 0.7) + 71$. The function $L(t) = 42 \sin(0.034t - 1.52) + 42$ models the rate at which people leave the fairgrounds. Both $A(t)$ and $L(t)$ are measured in people per minute and $t$ is measured for $0 \le t \le 180$ minutes. When the count begins at $t = 0$, there are already 1572 people in the flea market area of the fairgrounds.
   a. How many additional people arrive for the flea market during the 3-hour period after it opens to the public?
   b. Write an expression for $P(t)$, the total number of people at the flea market at time $t$.
   c. Find the value of $P'(75)$ and explain its meaning.
   d. For $0 \le t \le 180$, at what time is the rate of change of people at the flea market at a maximum? What is the maximum rate of change? Justify your answers.

## Part B
## Time: 45 minutes
## Number of Problems: 3

**You may not use a calculator for any problem in this section.**

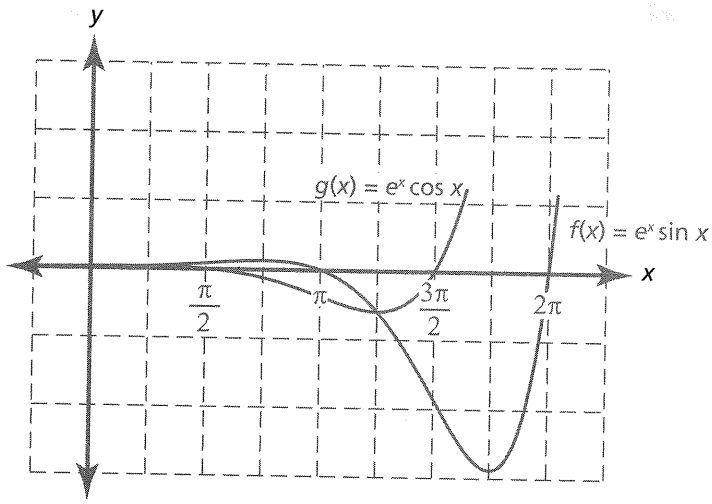
During the timed portion for Section II, Part B, you may continue to work on the problems in Part A without the use of a calculator.

4. Consider the differential equation $\frac{dy}{dx} = (y+1)(1-x)$ for $y > -1$.
   a. On the axes provided, sketch a slope field for the given differential equation at the 11 points indicated.
   b. If $y(0) = 1$, then find the particular solution $y(x)$ to the given differential equation.
   c. Draw a function through the point (0, 1) on your slope field which represents an approximate solution of the given differential equation with initial condition $y(0) = 1$.




5. Consider the function $h(x) = 3x^2 - \sqrt{x+1}$.
   a. Evaluate $\frac{1}{3-(-1)} \int_{-1}^{3} \left(3x^2 - \sqrt{x+1}\right) dx$ and interpret its meaning.
   b. What is the equation of the tangent to $h(x)$ at $x = 0$?
   c. Use the tangent found in part b to approximate $h(x)$ at $x = -0.01$.
   d. Is the approximation, found in part c, greater or less than the actual value of $h(x)$ at $x = -0.01$? Justify your answer using calculus.

6. Consider the functions $f(x) = e^x \sin x$ and $g(x) = e^x \cos x$, as shown in the sketch to the right, in the closed interval $0 \le x \le 2\pi$.



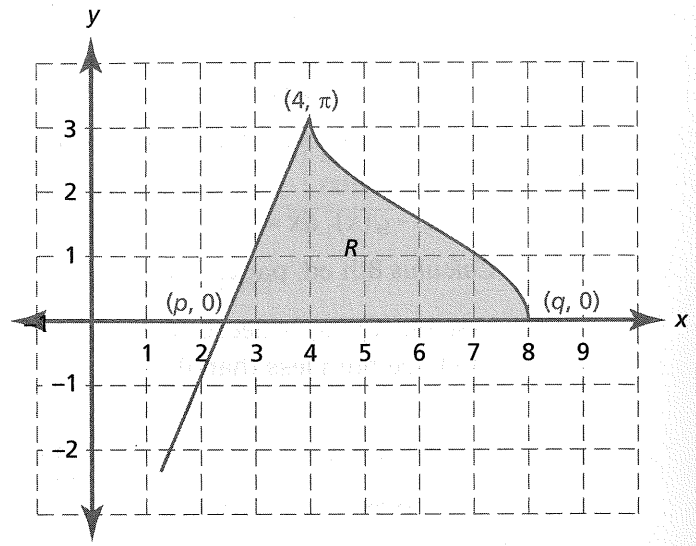

a. Let $D(x) = f(x) - g(x)$ be the vertical distance between the functions. Find the value of $x$ in the open interval $\frac{\pi}{4} < x < \frac{5\pi}{4}$, where $D(x)$ is a maximum value. Explain your reasoning.

b. At what value of $x$ in the open interval $\frac{\pi}{4} < x < \frac{5\pi}{4}$ is the rate of change of $D(x)$ increasing the most rapidly? Explain your reasoning.

c. For $H(x) = \frac{e^x}{\sin x} + g(x)$, find the $x$-coordinate of all points at which $H(x)$ has horizontal tangents on the open interval $0 < x < 2\pi$.

## FREE-RESPONSE QUESTIONS

1. Region $R$ is bounded by the functions $f(x) = 2(x-4)+\pi$, $g(x) = \cos^{-1}\left(\frac{\pi}{2}-3\right)$, and the $x$-axis as shown in the figure to the right.
   a. What is the area of region $R$?
   b. Find the volume of the solid generated when region $R$ is rotated about the $x$-axis.
   c. Find all values $c$ for $f(x)$ and $g(x)$ in the closed interval $p \le c \le q$ for which each function equals the average value in the indicated interval.




| | Solution | Possible points |
|---|---|---|
| a. | Root of $f(x)$ is $p = 2.4292$ and the root of $g(x)$ is $q = 8$.<br>Area of $R = \int_{2.4292}^{4} [2(x-4)+\pi]\,dx + \int_{4}^{8}\left[\cos^{-1}\left(\frac{x}{2}-3\right)\right]dx \approx 8.751$ | 3: $\begin{cases} 1: \text{limits and integrand for left branch, } f(x) \\ 1: \text{limits and integrand for right branch, } g(x) \\ 1: \text{answer} \end{cases}$ |
| b. | Volume =<br>$\pi\left\{\int_{2.4292}^{4}[2(x-4)+\pi]^2\,dx + \int_{4}^{8}\left[\cos^{-1}\left(\frac{x}{2}-3\right)\right]^2 dx\right\}$<br>$V \approx 16.907\pi$ or $53.115$ | 3: $\begin{cases} 1: \text{limits on both integrals} \\ 1: \text{both integrands} \\ \quad <-1> \text{ for any error} \\ 1: \text{answer} \end{cases}$ |
| c. | The average value of a function on an interval is given by $h(c) = \frac{1}{b-a}\int_a^b h(x)\,dx$, where $a \le c \le b$.<br>Use the answer from part a above.<br>For the left branch, $f(x)$:<br>$2(c_1-4)+\pi = \frac{1}{8-2.4292}[8.751]$<br>$\Rightarrow c_1 \approx 3.215$ $(p < c_1 < 4)$<br>For the right branch of $g(x)$:<br>$\cos^{-1}\left(\frac{c_2}{2}-3\right) = \frac{1}{8-2.4292}(8.751)$<br>$\Rightarrow c_2 = 6$ $(4 < c_2 < 8)$ | 3: $\begin{cases} 1: \text{use of } f_{avg} \\ 1: c_1 \text{ for left branch, } f(x) \\ 1: c_2 \text{ for right branch, } g(x) \end{cases}$ |

1. a (*Calculus* 8th ed. pages 446–455 / 9th ed. pages 448–457)
1. b (*Calculus* 8th ed. pages 456–466 / 9th ed. pages 458–468)
1. c (*Calculus* 8th ed. pages 282–294 / 9th ed. pages 282–296)

2. A particle is moving along the $x$-axis with velocity $v(t) = \ln(t+3) - e^{\frac{t}{2}-1}(\cos t)$, for $0 \le t \le 8$. The initial position of the particle is $-1.6$.
   a. At what time(s) in the open interval $0 < t < 8$ does the particle change direction? Justify your answer.
   b. Where is the particle when it is farthest to the left?
   c. How far does the particle travel in the interval $3 \le t \le 6$?
   d. At what times in the closed interval $0 \le t \le 8$ is the speed of the particle decreasing? Justify your answer.

<table>
<tr><th></th><th>Solution</th><th>Possible points</th></tr>
<tr><td>a.</td><td>The particle changes direction when $v(t)$ changes from positive to negative or negative to positive. Graph $v(t)$, then $v(t) = 0$ at $t \approx 5.160$ and $t \approx 7.718$.
<table>
<tr><td>$t$</td><td>$0 < t < 5.160$</td><td>$5.160 < t < 7.718$</td><td>$7.718 < t < 9$</td></tr>
<tr><td>$v(t)$</td><td>positive</td><td>negative</td><td>positive</td></tr>
</table>
The particle moves to the right from 0 to 5.160 sec, to the left from 5.160 to 7.718 sec, and to the right from 7.718 to 8 sec.</td><td>2: $\begin{cases} 1: & \text{values} \\ 1: & \text{reason} \end{cases}$</td></tr>
<tr><td>b.</td><td>$x(t) = x(0) + \int_0^{7.718} v(t)\,dt$<br>$x(7.718) \approx -1.6 - .2167 \approx -1.817$<br>The particle is 1.817 units to the left of the origin at $t = 7.718$.</td><td>2: $\begin{cases} 1: & \text{integrand with limits} \\ 1: & \text{answer, including } x(0) \end{cases}$</td></tr>
<tr><td>c.</td><td>Distance $= \int_3^6 |v(t)|\,dt \approx 8.455$</td><td>2: $\begin{cases} 1: & \text{integrand, with limits} \\ 1: & \text{answer} \end{cases}$</td></tr>
<tr><td>d.</td><td>The speed of the particle is decreasing when the velocity and acceleration have opposite signs.
<table>
<tr><th>$t$</th><th>$v(t)$</th><th>$a(t)$</th></tr>
<tr><td>$0 < t < 3.664$</td><td>positive</td><td>positive</td></tr>
<tr><td>$3.664 < t < 5.160$</td><td>positive</td><td>negative</td></tr>
<tr><td>$5.160 < t < 6.738$</td><td>negative</td><td>negative</td></tr>
<tr><td>$6.738 < t < 7.718$</td><td>negative</td><td>positive</td></tr>
<tr><td>$7.718 < t < 8$</td><td>positive</td><td>positive</td></tr>
</table>
Therefore, the speed of the particle is decreasing in the intervals $3.664 < t < 5.160$ and $6.738 < t < 7.718$.</td><td>3: $\begin{cases} 1: & v(t) \text{ and } a(t) \text{ have opposite signs} \\ 1: & \text{analysis} \\ 1: & \text{answers} \end{cases}$</td></tr>
</table>

2. a (*Calculus* 8th ed. pages 179–189 / 9th ed. pages 179–189)
2. b (*Calculus* 8th ed. pages 282–294 / 9th ed. pages 282–296)
2. c (*Calculus* 8th ed. pages 282–294 / 9th ed. pages 282–296)
2. d (*Calculus* 8th ed. pages 119–129 / 9th ed. pages 119–129)

3. A large flea market is held at the local fairgrounds on the first Saturday of each month. The rates at which people enter and leave the fairgrounds are recorded for a 3-hour period beginning when the market is open to the public. The rate at which people arrive is modeled by the function $A(t) = 45\sin(0.03t - .7) + 71$. The function $L(t) = 42\sin(0.034t - 1.52) + 42$ models the rate at which people leave the fairgrounds. Both $A(t)$ and $L(t)$ are measured in people per minute and $t$ is measured for $0 \le t \le 180$ minutes. When the count begins at $t = 0$, there are already 1572 people in the flea market area of the fairgrounds.
   a. How many additional people arrive for the flea market during the 3-hour period after it opens to the public?
   b. Write an expression for $P(t)$, the total number of people at the flea market at time $t$.
   c. Find the value of $P'(75)$ and explain its meaning.
   d. For $0 \le t \le 180$, at what time is the rate of change of people at the flea market at a maximum? What is the maximum rate of change? Justify your answers.

| | Solution | Possible points |
|---|---|---|
| a. | $\int_0^{180} A(t)\,dt = 13{,}945.84$<br>$\approx 13{,}945$ or $13{,}946$ people | 2: $\begin{cases} 1: \text{integral with limits} \\ 1: \text{answer} \end{cases}$ |
| b. | $P(t) = 1572 + \int_0^t [A(x) - L(x)]\,dx$ | 2: $\begin{cases} 1: \text{integral, with limit in terms of } t \\ 1: \text{answer includes } P(0) \text{ or } 1572 \end{cases}$ |
| c. | $P'(75)$ is the rate at which the number of people arriving and leaving the fairgrounds is changing 75 minutes after the flea market is open to the public.<br>$P'(t) = A(t) - L(t)$<br>$P'(75) = A(75) - L(75)$<br>$P'(75) \approx 115.990 - 78.007 = 37.984$<br>The rate is approximately 37 or 38 people per minute and the number of people at the flea market is increasing because $P'(75) > 0$. | 2: $\begin{cases} 1: \text{explanation} \\ 1: \text{value of } P'(75) \end{cases}$ |
| d. | To maximize $P'(t)$,<br>$P''(t) = A'(t) - L'(t) = 0 \Rightarrow t \approx 32.255$ or $127.319$.<br>Since the interval is closed, check the value of $P'(t)$ at these two values as well as at the end points of the closed interval.<br>$t$: 0, 32.255, 127.319, 180<br>$P'(t)$: 41.956, 58.155 maximum, 16.272 minimum, 25.738<br>The maximum rate is 58 or 59 people per minute at $t = 32.255$ minutes. | 3: $\begin{cases} 1: t = 32.255 \\ 1: P'(32.255) \\ 1: \text{reason} \end{cases}$ |

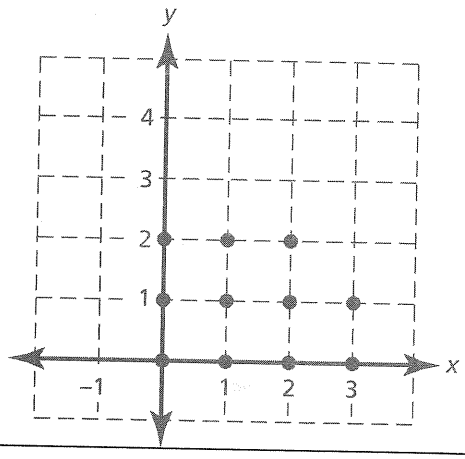
3. a, b, c (*Calculus* 8th ed. pages 282–294 / 9th ed. pages 282–296)
3. d (*Calculus* 8th ed. pages 179–189 / 9th ed. pages 179–189)

4. Consider the differential equation $\frac{dy}{dx} = (y+1)(1-x)$ for $y > -1$.

   a. On the axes provided, sketch a slope field for the given differential equation at the 11 points indicated.

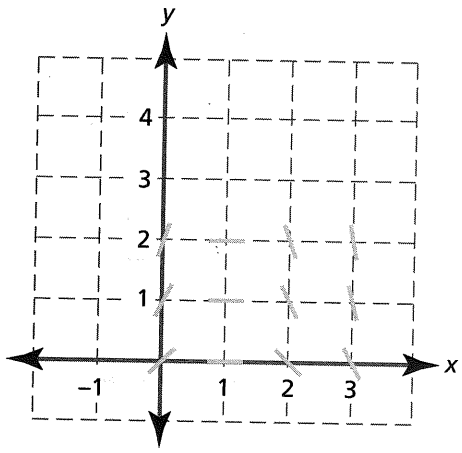
   b. If $y(0) = 1$, then find the particular solution $y(x)$ to the given differential equation.

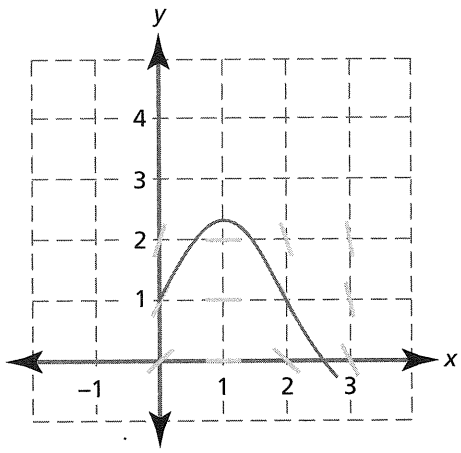
   c. Draw a function through the point (0, 1) on your slope field which represents an approximate solution of the given differential equation with initial condition $y(0) = 1$.




| | Solution | Possible points |
|---|---|---|
| a. | The values of $dy/dx$ at the points indicated are given in the table and the sketch below. | 2: 1: zero slope at each point $(x, y)$ where $x = 1$; 1: positive slopes at each point $(x, y)$ where $x = 0$, negative slopes at each point $(x, y)$ where $x = 2$ or 3 |
| b. | $\frac{dy}{y+1} = (1-x)\,dx \Rightarrow$ <br> $\ln\lvert y+1\rvert = x - \frac{x^2}{2} + c_1$. Recall that $y > -1$, so <br> $y+1 = e^{x-\frac{x^2}{2}+c_1} = e^{x-\frac{x^2}{2}}e^{c_1} = Ce^{x-\frac{x^2}{2}}$. <br> Then, $y = -1 + Ce^{x-\frac{x^2}{2}}$. Using the initial condition (0, 1), $C = 2$ and $y = -1 + 2e^{x-\frac{x^2}{2}}$. | 5: 1: separating variables; 1: antiderivatives; 1: constant of integration; 1: uses initial condition; 1: solves for $y$, 0/1 if $y$ is not exponential |

| $y \backslash x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 2 | 3 | 0 | −3 | −6 |
| 1 | 2 | 0 | −2 | −4 |
| 0 | 1 | 0 | −1 | −2 |



| | Solution | Possible points |
|---|---|---|
| c. |  <br>$y = -1 + 2e^{x - \frac{x^2}{2}}$ | 2: { 1: maximum at $x = 1$ only curve above $x$-axis; 1: following slopes from part a } |

4. a, b, c (*Calculus* 8th ed. pages 404–408 / 9th ed. pages 406–410)

5. Consider the function $h(x) = 3x^2 - \sqrt{x+1}$.

   a. Evaluate $\frac{1}{3-(-1)}\int_{-1}^{3}\left(3x^2 - \sqrt{x+1}\right)dx$ and interpret its meaning.

   b. What is the equation of the tangent to $h(x)$ at $x = 0$?

   c. Use the tangent found in part b to approximate $h(x)$ at $x = -0.01$.

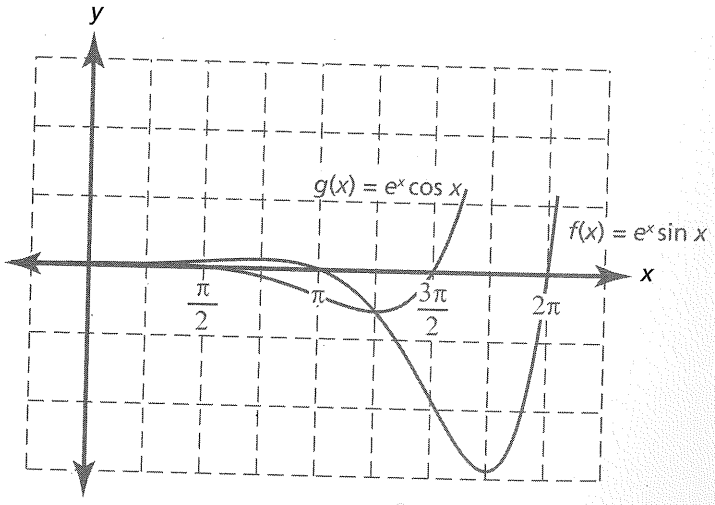
   d. Is the approximation, found in part c, greater or less than the actual value of $h(x)$ at $x = -0.01$? Justify your answer using calculus.

| | Solution | Possible points |
|---|---|---|
| a. | Rewrite the given integral:<br>$\frac{1}{4}\int_{-1}^{3}\left(3x^2 - (x+1)^{\frac{1}{2}}\right)dx$<br>$= \frac{1}{4}\left\{x^3 - \left(\frac{2}{3}\right)\left[(x+1)^{\frac{3}{2}}\right]\right\}\Big\vert_{-1}^{3}$<br>$= \frac{1}{4}\left[27 - \frac{2}{3}(8) - (-1-0)\right]$<br>$= \frac{17}{3}$<br>The integral represents the average value of the function in the interval $-1 \le x \le 3$. | 4: { 2: antiderivatives <−1> for each error; 1: value of integral expression; 1: recognizing average value of $h(x)$ on interval $-1 \le x \le 3$ } |

| | Solution | Possible points |
|---|---|---|
| b. | $h(x) = 3x^2 - (x+1)^{\frac{1}{2}}$ and $h(0) = -1$<br>$h'(x) = 6x - \left[\left(\frac{1}{2}\right)(x+1)^{-\frac{1}{2}}\right]$<br>$h'(0) = 0 - \frac{1}{2}(1)^{-\frac{1}{2}} = -\frac{1}{2}$<br>Since $h(0) = 1$ and $h'(0) = -\frac{1}{2}$, then the tangent line to $h(x)$ at $x = 0$ is<br>$y + 1 = -\frac{1}{2}(x) \Rightarrow y = -\frac{1}{2}(x) - 1$. | 2: $\begin{cases} 1: & h'(0) \\ 1: & \text{equation of tangent line} \end{cases}$ |
| c. | $h(-.01) \approx -\frac{1}{2}(-0.01) - 1 = -0.995$ | 1: answer |
| d. | $h''(x) = 6 + \frac{1}{4}(x+1)^{-\frac{3}{2}}$ and $h''(x) > 0$ for all $x$. Therefore $h(x)$ is concave up at $x = 0$ and the tangent line will be below the curve. Thus the value of the approximation near $x = 0$ will be less than the actual value. | 2: $\begin{cases} 1: & \text{finding } h''(x) \\ 1: & \text{reason for under approximation} \end{cases}$ |

5. a (*Calculus* 8th ed. pages 282–294 / 9th ed. pages 282–296)
5. b (*Calculus* 8th ed. pages 130–140 / 9th ed. pages 130–140)
5. c, d (*Calculus* 8th ed. pages 235–247 / 9th ed. pages 235–247)

6. Consider the functions $f(x) = e^x \sin x$ and $g(x) = e^x \cos x$, as shown in the sketch to the right, in the closed interval $0 \le x \le 2\pi$.




a. Let $D(x) = f(x) - g(x)$ be the vertical distance between the functions. Find the value of $x$ in the open interval $\frac{\pi}{4} < x < \frac{5\pi}{4}$, where $D(x)$ is a maximum value. Explain your reasoning.

b. At what value of $x$ in the open interval $\frac{\pi}{4} < x < \frac{5\pi}{4}$ is the rate of change of $D(x)$ increasing the most rapidly? Explain your reasoning.

c. For $H(x) = \frac{e^x}{\sin x} + g(x)$, find the $x$-coordinate of all points at which $H(x)$ has horizontal tangents on the open interval $0 < x < 2\pi$.

<table>
<tr><th></th><th>Solution</th><th>Possible points</th></tr>
<tr><td>a.</td><td>$D(x) = f(x) - g(x)$<br>$D'(x) = f'(x) - g'(x)$<br>$D'(x) = e^x \sin x + e^x \cos x - (e^x \cos x - e^x \sin x)$<br>$= 2e^x \sin x = 0 \Rightarrow x = \pi$<br><table><tr><th>$x$</th><th>$\frac{\pi}{4} < x < \pi$</th><th>$\pi < x < \frac{5\pi}{4}$</th></tr><tr><td>$D'(x)$</td><td>positive</td><td>negative</td></tr><tr><td>$D(x)$</td><td>increasing</td><td>decreasing</td></tr></table>Therefore, $D(x)$ is a maximum at $x = \pi$, because $D(x)$ is increasing to the left of $x = \pi$ and decreasing to the right of $x = \pi$, which is the only critical value in the interval $\frac{\pi}{4} < x < \frac{5\pi}{4}$.</td><td>3: $\begin{cases} 1: & \text{find } D'(x) \\ 1: & \text{set } D'(x) = 0 \\ 1: & \text{answer and reason} \end{cases}$</td></tr>
<tr><td>b.</td><td>The rate of change of $D(x)$ is given by $D'(x)$, and the absolute maximum of $D'(x)$ occurs when $D''(x) = 0$.<br>$D''(x) = 2e^x \sin x + 2e^x \cos x = 0$<br>$= 2e^x(\sin x + \cos x) = 0$<br>$\Rightarrow \sin x = -\cos x \Rightarrow x = \frac{3\pi}{4}$.<br><table><tr><th>$x$</th><th>$\frac{\pi}{4} < x < \frac{3\pi}{4}$</th><th>$\frac{3\pi}{4} < x < \frac{5\pi}{4}$</th></tr><tr><td>$D''(x)$</td><td>positive</td><td>negative</td></tr><tr><td>$D'(x)$</td><td>increasing</td><td>decreasing</td></tr></table>Therefore, $D'(x)$, the rate of change in the distance, is increasing most rapidly (an absolute maximum) at $x = \frac{3\pi}{4}$ because $D'(x)$ is increasing to the left of $x = \frac{3\pi}{4}$ and decreasing to the right of $x = \frac{3\pi}{4}$, which is the only critical value in the interval $\frac{\pi}{4} < x < \frac{5\pi}{4}$.</td><td>3: $\begin{cases} 1: & \text{find } D''(x) \\ 1: & \text{set } D''(x) = 0 \\ 1: & \text{answer with reason} \end{cases}$</td></tr>
</table>

| | Solution | Possible points |
|---|---|---|
| c. | $H(x)=\dfrac{e^x}{\sin x}+g(x)$<br>$H'(x)=\dfrac{e^x\sin x-e^x\cos x}{\sin^2 x}+e^x\cos x-e^x\sin x$<br>$\Rightarrow \dfrac{e^x(\sin x-\cos x)-e^x\sin^2 x(\sin x-\cos x)}{\sin^2 x}$<br>$\Rightarrow \dfrac{e^x(\sin x-\cos x)(1-\sin^2 x)}{\sin^2 x}=0$<br>In the open interval $0<x<2\pi$, the horizontal tangents occur when $H'(x)=0$. Then,<br>$1-\sin^2 x=0 \Rightarrow x=\dfrac{\pi}{2}$ and $\dfrac{3\pi}{2}$, or<br>$\sin x-\cos x=0 \Rightarrow x=\dfrac{\pi}{4}$ and $\dfrac{5\pi}{4}$. | 3: $\begin{cases}1: & \text{find derivative of } H(x)\\ 1: & \text{set } H'(x)=0\\ 1: & \text{answers}\end{cases}$ |

6. a (*Calculus* 8th ed. pages 179–189 / 9th ed. pages 179–189)
6. b (*Calculus* 8th ed. pages 190–197 / 9th ed. pages 190–197)
6. c (*Calculus* 8th ed. pages 119–129 / 9th ed. pages 119–129)