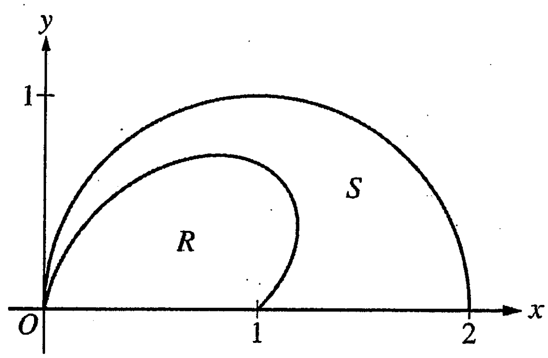
AB/BC Cirque Extra Credit Spring 2018 Assignment #2:

Directions: Create and Solve 1 original FRQ. Then answer the following 4 FRQs, make corrections (red ink) and score the FRQs. Due Mon(5/14)

1)



The figure above shows the polar curves $r = f(\theta) = 1 + \sin\theta\cos(2\theta)$ and $r = g(\theta) = 2\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$. Let R be the region in the first quadrant bounded by the curve $r = f(\theta)$ and the x-axis. Let S be the region in the first quadrant bounded by the curve $r = f(\theta)$, the curve $r = g(\theta)$, and the x-axis.

(a) Find the area of R.

(b) The ray $\theta = k$, where $0 < k < \frac{\pi}{2}$, divides S into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of k.

(c) For each θ, $0 \le \theta \le \frac{\pi}{2}$, let $w(\theta)$ be the distance between the points with polar coordinates $(f(\theta), \theta)$ and $(g(\theta), \theta)$. Write an expression for $w(\theta)$. Find w_A, the average value of $w(\theta)$ over the interval $0 \le \theta \le \frac{\pi}{2}$.

(d) Using the information from part (c), find the value of θ for which $w(\theta) = w_A$. Is the function $w(\theta)$ increasing or decreasing at that value of θ? Give a reason for your answer.

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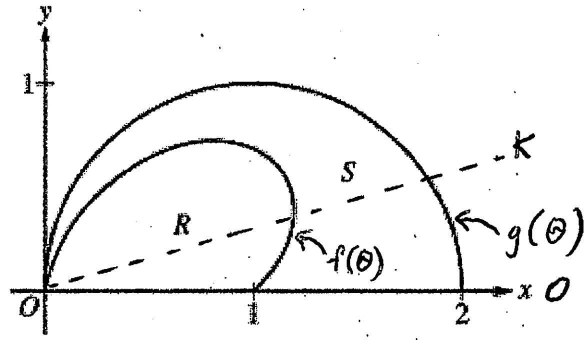

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[2] a) Area of $R = \frac{1}{2}\int_0^{\pi/2} [f(\theta)]^2 d\theta = \frac{1}{2}\int_0^{\pi/2} [1 + \sin\theta\cos 2\theta]^2 d\theta = \boxed{0.648}$

[3] b) Region $S = \frac{1}{2}\int_0^{\pi/2} g(\theta)^2 d\theta - \frac{1}{2}\int_0^{\pi/2} f(\theta)^2 d\theta$
$= \frac{1}{2}\int_0^{\pi/2} g(\theta)^2 - f(\theta)^2 d\theta$

2 equal regions of S: $\frac{1}{2}\int_0^{K} g(\theta)^2 - f(\theta)^2 d\theta = \frac{1}{2}\int_K^{\pi/2} g(\theta)^2 - f(\theta)^2 d\theta$

[2] c) $w(\theta) = g(\theta) - f(\theta)$

*Avg. value theorem: $\frac{1}{b-a}\int_a^b w(\theta)d\theta = \frac{1}{\pi/2 - 0}\int_0^{\pi/2} g(\theta) - f(\theta) d\theta = \frac{2}{\pi}\int_0^{\pi/2} g(\theta) - f(\theta) d\theta = \boxed{0.485}$

[2] d) *set $g(\theta) - f(\theta) = 0.485$

*look for intersection: $\theta = 0.517$

$w'(0.517) = -0.581$

$w(\theta)$ is decreasing since $w'(0.518) < 0$

2)

Question

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	−4	4	2
4	−1	3	6	7

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

(c) Let w be the function given by $w(x) = \int_1^{g(x)} f(t)\,dt$. Find the value of $w'(3)$.

(d) If g^{-1} is the inverse function of g, write an equation for the line tangent to the graph of $y = g^{-1}(x)$ at $x = 2$.

2)

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(a) $h(1) = f(g(1)) - 6 = f(2) - 6 = 9 - 6 = 3$
$h(3) = f(g(3)) - 6 = f(4) - 6 = -1 - 6 = -7$
Since $h(3) < -5 < h(1)$ and h is continuous, by the Intermediate Value Theorem, there exists a value r, $1 < r < 3$, such that $h(r) = -5$.

$2: \begin{cases} 1: h(1) \text{ and } h(3) \\ 1: \text{conclusion, using IVT} \end{cases}$

(b) $\dfrac{h(3) - h(1)}{3 - 1} = \dfrac{-7 - 3}{3 - 1} = -5$
Since h is continuous and differentiable, by the Mean Value Theorem, there exists a value c, $1 < c < 3$, such that $h'(c) = -5$.

$2: \begin{cases} 1: \dfrac{h(3) - h(1)}{3 - 1} \\ 1: \text{conclusion, using MVT} \end{cases}$

(c) $w'(3) = f(g(3)) \cdot g'(3) = f(4) \cdot 2 = -2$

$2: \begin{cases} 1: \text{apply chain rule} \\ 1: \text{answer} \end{cases}$

(d) $g(1) = 2$, so $g^{-1}(2) = 1$.
$$\left(g^{-1}\right)'(2) = \frac{1}{g'\left(g^{-1}(2)\right)} = \frac{1}{g'(1)} = \frac{1}{5}$$
An equation of the tangent line is $y - 1 = \frac{1}{5}(x - 2)$.

$3: \begin{cases} 1: g^{-1}(2) \\ 1: \left(g^{-1}\right)'(2) \\ 1: \text{tangent line equation} \end{cases}$

3)

x	-1.5	-1.0	-0.5	0	0.5	1.0	1.5
$f(x)$	-1	-4	-6	-7	-6	-4	-1
$f'(x)$	-7	-5	-3	0	3	5	7

Let f be a function that is differentiable for all real numbers. The table above gives the values of f and its derivative f' for selected points x in the closed interval $-1.5 \le x \le 1.5$. The second derivative of f has the property that $f''(x) > 0$ for $-1.5 \le x \le 1.5$.

(a) Evaluate $\int_0^{1.5} (3f'(x) + 4)\,dx$. Show the work that leads to your answer.

(b) Write an equation of the line tangent to the graph of f at the point where $x = 1$. Use this line to approximate the value of $f(1.2)$. Is this approximation greater than or less than the actual value of $f(1.2)$? Give a reason for your answer.

(c) Find a positive real number r having the property that there must exist a value c with $0 < c < 0.5$ and $f''(c) = r$. Give a reason for your answer.

(d) Let g be the function given by $g(x) = \begin{cases} 2x^2 - x - 7 & \text{for } x < 0 \\ 2x^2 + x - 7 & \text{for } x \ge 0. \end{cases}$

The graph of g passes through each of the points $(x, f(x))$ given in the table above. Is it possible that f and g are the same function? Give a reason for your answer.

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The graph of g passes through each of the points $(x, f(x))$ given in the table above. Is it possible that f and g are the same function? Give a reason for your answer.

(a) $\int_0^{1.5} (3f'(x) + 4)\,dx = 3\int_0^{1.5} f'(x)\,dx + \int_0^{1.5} 4\,dx$

$= 3f(x) + 4x \Big|_0^{1.5} = 3(-1 - (-7)) + 4(1.5) = 24$

2 { 1 : antiderivative; 1 : answer }

(b) $y = 5(x - 1) - 4$

$f(1.2) \approx 5(0.2) - 4 = -3$

The approximation is less than $f(1.2)$ because the graph of f is concave up on the interval $1 < x < 1.2$.

3 { 1 : tangent line; 1 : computes y on tangent line at $x = 1.2$; 1 : answer with reason }

(c) By the Mean Value Theorem there is a c with $0 < c < 0.5$ such that

$$f''(c) = \frac{f'(0.5) - f'(0)}{0.5 - 0} = \frac{3 - 0}{0.5} = 6 = r$$

2 { 1 : reference to MVT for f' (or differentiability of f'); 1 : value of r for interval $0 \le x \le 0.5$ }

(d) $\lim_{x \to 0^-} g'(x) = \lim_{x \to 0^-} (4x - 1) = -1$

$\lim_{x \to 0^+} g'(x) = \lim_{x \to 0^+} (4x + 1) = +1$

Thus g' is not continuous at $x = 0$, but f' is continuous at $x = 0$, so $f \ne g$.

OR

$g''(x) = 4$ for all $x \ne 0$, but it was shown in part (c) that $f''(c) = 6$ for some $c \ne 0$, so $f \ne g$.

2 { 1 : answers "no" with reference to g' or g''; 1 : correct reason }

BC # 1

4) The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

(c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

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[2] a) $f''(1) = \frac{(-1)^2(2-1)!}{2^2} = \frac{1}{4}$

$f^3(1) = \frac{(-1)^3(3-1)!}{2^3} = \frac{-2}{8} = \frac{-1}{4}$

* Taylor Series : $\frac{f^n(c)}{n!}(x-c)^n$

$f(x) = 1 - \frac{1}{2}(x-1) + \frac{\frac{1}{4}}{2!}(x-1)^2 - \frac{\frac{1}{4}}{3!}(x-1)^3$

first 4 nonzero terms = $1 - \frac{1}{2}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{24}(x-1)^3 + \frac{(-1)^n}{n \cdot 2^n}(x-1)^n$ ← General Term

$* \frac{\frac{(-1)^n(n-1)!}{2^n}(x-1)^n}{n!} \cdot \frac{(-1)^n(n-1)!}{2^n \cdot n!} = \frac{(-1)^n}{2^n \cdot n!}$

[3] b) $|x-c| = r$ radius = 2, center = 1

$|x-1| < 2$

$-2 < x-1 < 2$

$-1 < x < 3$

Test endpoints at $x = -1, 3$

$\sum \frac{(-1)^n}{n \cdot 2^n}(x-1)^n$

At $x=-1$, $\sum \frac{(-1)^n}{2^n(n)}(-1-1)^n = \frac{(-1)^n(-2)^n}{2^n \cdot n} = \frac{1}{n}$

diverges by harmonic series

At $x=3$, $\sum \frac{(-1)^n}{2^n \cdot n}(3-1)^n = \frac{(-1)^n(2^n)}{2^n \cdot n} = \frac{(-1)^n}{n}$

Converges by Alt. series Test.

Interval of convergence $(-1, 3]$

[2] c) $f(1.2) \approx 1 - \frac{1}{2}(1.2-1) + \frac{1}{8}(1.2-1)^2$

$= 0.905$

[2] d) By Alternating series remainder, the error of this approximation is the first unused (neglected) term

first unused term → $\frac{1}{24}(x-1)^3 = \frac{1}{24}(1.2-1)^3 = \frac{1}{3000} < 0.001$