A.P. Calculus AB **Review for Test on Integrals of Logs/Exps (Chapter 6, 7, Slope fields)**

Calculators permitted.

1. Oil is being pumped continuously from a certain oil well at a rate proportional to the amount of oil left in the well; that is, $\frac{dy}{dt} = ky$, where y is the amount of oil left in the well at any time t. Initially there were 1,000,000 gallons of oil in the well, and 6 years later there were 500,000 gallons remaining.

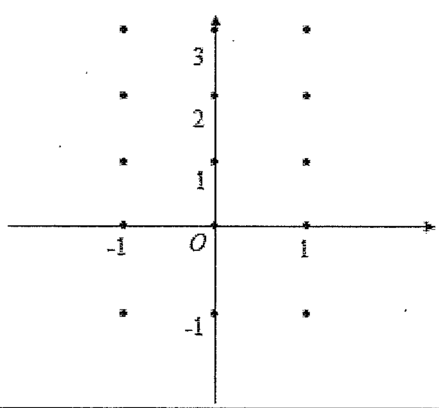
a. Write an equation for y, the amount of oil remaining in the well at any time t.

b. At what rate is the amount of oil in the well decreasing when there are 600,000 gallons of oil remaining?

c. It will no longer be profitable to pump oil when there are fewer than 50,000 gallons remaining. In order not to lose money, at what time t should oil no longer be pumped from the well?

2. Consider the differential equation $\frac{dy}{dx} = x^2(y-1)$.

a) On the axes below, sketch a slope field for the given differential equation at the fifteen points indicated.



b) While the slope field in part a is drawn at only fifteen points, it is defined at every point in the xy-plane. Describe all points in the xy-plane for which the slopes are negative.

c) Find the particular solution, $y = f(x)$ to the given diff. equation with the initial condition $f(0) = 3$.

d) Find the equation of the line tangent to $y = f(x)$ at the point where $x = 0$ and use it to approximate $f(0.1)$.

3. Let $v(t)$ be the velocity, in feet per second, of a skydiver at time t seconds, $t \geq 0$. After her parachute opens, her velocity satisfies the differential equation $\frac{dv}{dt} = -2(v+16)$, with initial condition $v(0) = -50$.

a) Use separation of variables to find an expression for v in terms of t, where t is measured in seconds.

b) Terminal velocity is defined as $\lim_{t \to \infty} v(t)$. Find the terminal velocity of the skydiver to the nearest foot per second.

c) It is safe to land when her <u>speed</u> is 20 feet per second. At what time, t, does she reach this speed?

4. Evaluate $\int \frac{x^2}{\sqrt{1-x^6}}dx$

5. $\int \frac{-4}{x^2 - 6x + 14} dx$

6. $\int \frac{-\csc(2x)\cot(2x)}{9^{\csc(2x)}} dx$

7. $\int sec\left(\frac{x}{2}\right)$dx

A.P. Calculus AB **Review for Test on Integrals of Logs/Exps** Solution Key

Calculators permitted.

1. Oil is being pumped continuously from a certain oil well at a rate proportional to the amount of oil left in the well; that is, $\frac{dy}{dt} = ky$, where y is the amount of oil left in the well at any time t. Initially there were 1,000,000 gallons of oil in the well, and 6 years later there were 500,000 gallons remaining. (time, oil) (t, y)

$(0, 1{,}000{,}000)$ $(6, 500{,}000)$

a) Write an equation for y, the amount of oil remaining in the well at any time t.

$\int \frac{dy}{y} = \int k\,dt$

$\ln|y| = kt + C$

$y = e^{kt} \cdot e^{C}$

$y = Ce^{kt}$

$1{,}000{,}000 = Ce^{k(0)}$

$1{,}000{,}000 = C$

$y = 1{,}000{,}000e^{kt}$

$500{,}000 = 1{,}000{,}000e^{6k}$

$\frac{1}{2} = e^{6k}$

$\ln 0.5 = \ln e^{6k}$

$\ln 0.5 = 6k$

$\frac{1}{6}\ln 0.5 = k$

$y = 1000000e^{\frac{1}{6}\ln 0.5 t}$

b. At what rate is the amount of oil in the well decreasing when there are 600,000 gallons of oil remaining?

Find $\frac{dy}{dt}$ when $y = 600{,}000$

$\frac{dy}{dt} = ky$

$\frac{dy}{dt} = \left(\frac{1}{6}\ln 0.5\right)(600{,}000) = -69314.718$ gallons/yr.

b) It will no longer be profitable to pump oil when there are fewer than 50,000 gallons remaining. In order not to lose money, at what time t should oil no longer be pumped from the well?

$(t, 50{,}000)$

$\frac{50{,}000}{1000000} = \frac{1000000e^{\frac{1}{6}\ln 0.5 t}}{1000000}$

$0.05 = e^{\frac{1}{6}\ln 0.5 t}$

$\ln 0.05 = \ln e^{\frac{1}{6}\ln 0.5 t}$

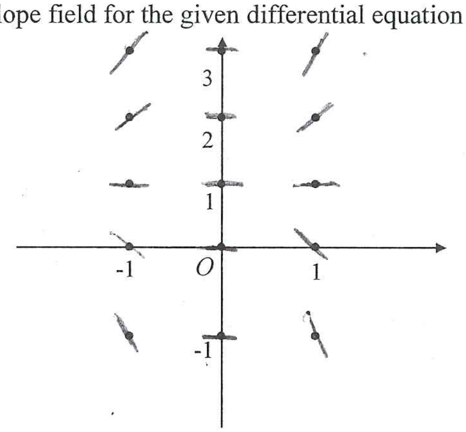
$\ln 0.05 = \frac{1}{6}\ln 0.5 t$

$\frac{6\ln 0.05}{\ln 0.5} = t$

$\boxed{t \approx 25.932 \text{ yrs. later}}$

2. Consider the differential equation $\frac{dy}{dx} = x^2(y-1)$.

a) On the axes below, sketch a slope field for the given differential equation at the fifteen points indicated.



b) While the slope field in part a is drawn at only fifteen points, it is defined at every point in the xy-plane. Describe all points in the xy-plane for which the slopes are negative.

Since x^2 is always positive, the determining factor is the y-value. When $y < 1$, then $\frac{dy}{dx} < 0$. Basically, all points below the line $y=1$ will have negative slope

c) Find the particular solution, $y = f(x)$ to the given diff. equation with the initial condition $f(0) = 3$.

$\int \frac{dy}{y-1} = \int x^2\,dx$

$e^{\ln|y-1|} = e^{\frac{x^3}{3} + C}$

$y - 1 = e^{x^3/3} \cdot e^{C}$

$y - 1 = Ce^{x^3/3}$

$3 - 1 = Ce^{0/3}$

$2 = C$

$y - 1 = 2e^{x^3/3}$

$\boxed{y = 2e^{x^3/3} + 1}$

d) Find the equation of the line tangent to $y = f(x)$ at the point where $x = 0$ and use it to approximate $f(0.1)$.

Find ordered pair: $(0,3)$

$y = 2e^{0/3} + 1$

$y = 2(1) + 1 = 3$

Find slope: $m = 0$

$\frac{dy}{dx} = x^2(y-1)$

$\left.\frac{dy}{dx}\right|_{(0,3)} = 0^2(3-1) = 0$

$y - y_1 = m(x - x_1)$

$y - 3 = 0(x - 0)$

$y = 3$

$\boxed{y(0.1) = 3}$

3. Let $v(t)$ be the velocity, in feet per second, of a skydiver at time t seconds, $t \geq 0$. After her parachute opens, her velocity satisfies the differential equation $\frac{dv}{dt} = -2(v+16)$, with initial condition $v(0) = -50$.

a) Use separation of variables to find an expression for v in terms of t, where t is measured in seconds.

$\int \frac{dv}{v+16} = \int -2dt$

$\ln|v+16| = -2t + c$

$e \qquad e$

$v+16 = e^{-2t} \cdot e^{c}$

$v+16 = Ce^{-2t}$

(time, velocity)

$(0, -50)$

$-50+16 = Ce^{-2(0)}$

$-34 = C$

$v+16 = -34e^{-2t}$

$\boxed{v(t) = -34e^{-2t} - 16}$

b) Terminal velocity is defined as $\lim_{t\to\infty} v(t)$. Find the terminal velocity of the skydiver to the nearest foot per second.

$\lim_{t\to\infty} v(t)$

$\lim_{t\to\infty} -34e^{-2t} - 16$

$\lim_{t\to\infty} \frac{-34}{e^{2t}} - 16$

$= \frac{-34}{e^{2(\infty)}} - 16 = 0 - 16 = \boxed{-16 \text{ ft/s}}$

c) It is safe to land when her <u>speed</u> is 20 feet per second. At what time, t, does she reach this speed?

$(t, -20)$

The skydiver is falling, so speed of 20ft/s is in this situation $v(t) = -20$ ft/s

$v(t) = -34e^{-2t} - 16$

$-20 = -34e^{-2t} - 16$

$-4 = -34e^{-2t}$

$\frac{4}{34} = e^{-2t}$

$\frac{2}{17} = e^{-2t}$

$\ln(2/17) = \ln e^{-2t}$

$\ln(2/17) = -2t$

$-\frac{1}{2}\ln(2/17) = t$

$\boxed{t = 1.07 \text{ seconds}}$

4. Evaluate $\int \frac{x^2}{\sqrt{1-x^6}}dx$

$\int \frac{x^2}{\sqrt{(1)^2-(x^3)^2}}dx$

$a=1$

$u=x^3$

$\frac{du}{dx} = 3x^2$

$dx = \frac{du}{3x^2}$

$= \int \frac{\cancel{x^2}}{\sqrt{a^2-u^2}} \cdot \frac{du}{3\cancel{x^2}}$

$* \int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + c$

$= \frac{1}{3}\int \frac{du}{\sqrt{a^2-u^2}}$

$= \frac{1}{3}\arcsin\left(\frac{u}{a}\right) + c$

$= \boxed{\frac{1}{3}\arcsin(x^3) + C}$

5. $\int \frac{-4}{x^2 - 6x + 14}dx$ *complete the square

$x^2 - 6x + \_ + 14 - \_$

$\left(\frac{b}{2}\right)^2 = \left(\frac{6}{2}\right)^2 = 9$

$x^2 - 6x + 9 + 14 - 9$

$(x-3)^2 + 5$

$\int \frac{-4}{(x-3)^2 + (\sqrt{5})^2}dx$

$u = x-3 \quad dx = du$

$\frac{du}{dx} = 1 \quad a = \sqrt{5}$

$\int \frac{-4}{u^2+a^2}du$ *arctan rule

$= -4 \cdot \frac{1}{a}\arctan\left(\frac{u}{a}\right) + c$

$= -4 \cdot \frac{1}{\sqrt{5}}\arctan\left(\frac{x-3}{\sqrt{5}}\right) + c$

$= \boxed{\frac{-4}{\sqrt{5}}\arctan\left(\frac{x-3}{\sqrt{5}}\right) + C}$

6. $\int \frac{-\csc(2x)\cot(2x)}{9^{\csc(2x)}}dx$ *Recall $\int a^u du = \frac{a^u}{\ln a} + C$

$\int 9^{-\csc(2x)} \cdot (-\csc(2x)\cot(2x))dx$

$u = -\csc(2x)$

$\frac{du}{dx} = -\csc(2x)\cot(2x) \cdot 2$

$dx = \frac{du}{-2\csc(2x)\cot(2x)}$

$\int 9^u \cdot \cancel{-\csc(2x)\cot(2x)} \cdot \frac{du}{-2\cancel{\csc(2x)\cot(2x)}}$

$+\frac{1}{2}\int 9^u du = +\frac{1}{2} \cdot \frac{9^u}{\ln 9} + C = \boxed{\frac{+9^{-\csc 2x}}{2\ln 9} + C}$

7. $\int \sec\left(\frac{x}{2}\right)dx$

*Recall: $\int \sec u\, du = \ln|\sec u + \tan u| + C$

$u = \frac{x}{2} \quad \frac{du}{dx} = \frac{1}{2} \quad dx = 2du$

$\int \sec u \cdot 2du = 2\int \sec u\, du$

$= 2\ln|\sec u + \tan u| + C$

$\boxed{= 2\ln\left|\sec\left(\frac{x}{2}\right) + \tan\left(\frac{x}{2}\right)\right| + C}$